THE CALCULUS TOOLKIT™

FOR APPLE® II PLUS, IIe, AND IIc COMPUTERS

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USER MANUAL

THE CALCULUS TOOLKIT™

For Apple® II Plus, IIe, and IIc Computers*

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EQUIPMENT NEEDED

Apple II Plus, IIe, or IIc microcomputer
One disk drive
48K memory
Monitor (color capability optional)

STARTING UP

1. Insert a program disk into the disk drive. The label on the disk should be facing *up* and the oval cutout pointing *inward.* Make sure that the diskette is completely inside before closing the door.

2. Turn on the monitor and the computer. The red light on the disk drive will glow and the drive will be activated. After 10 seconds, the screen will show the Addison-Wesley triangular logo.

3. Allow 15 seconds for the screen to display a list of programs from which to make a selection. (*Note:* You can reduce the waiting time by pressing the space bar to reach the program list.) If your computer is an Apple IIc or IIe, check to make sure that the Caps Lock key is depressed at this point.

4. If the program list fails to appear, check to see if your computer and monitor are properly connected. Should the disk still fail to operate, try one of the following procedures.

HELP

1. If the red light on the disk drive has gone out, open the disk drive door, remove the disk and turn off the computer; then re-insert the disk, close the door, and turn the computer back on. If the disk still fails to operate, try another disk.

2. If the red light stays on and the disk drive continues to spin, press the Control key and hold it down while pressing the Reset key. If the red light then goes out, follow Procedure 1 above. If the red light stays on, leave the disk drive running and get help.

HANDLING DISKETTES

Each diskette is encased in black plastic to minimize the amount of surface exposed to the drives. To protect this sensitive electromagnetic surface, please observe the following rules:

1. Never turn the computer off when the disk drive's red light is on. If you do, you will probably damage the diskette.

2. Put the disk into the drive before you turn the computer on.

3. Remove the diskette before you turn the computer off.

4. Grasp the diskette by the label, to avoid touching the surface through the holes. Scratched disks don't work.

5. Use felt tip pens to write on disk labels; never use pencils or pens.

6. Replace diskettes in their protective wrappers when not in use. Avoid leaving exposed diskettes on a table.

7. Store diskettes away from heat and strong sunlight.

8. Keep disks away from magnets, such as those frequently used at copystands. Exposure to magnetic fields of the kind generated by power transformers, for example, can damage the programs permanently.

Contents

The following authors created the individual programs of *The Calculus Toolkit:*

Dale T. Hoffman

A. Super * Grapher
B. Name That Function
C. Secant Lines
D. Limit Problems
E. Limit Definition
F. Continuity at a Point
G. Derivative Grapher
H. Function Evaluator/Comparer
O. Partial Fraction Integration Problems
Q. Sequences and Series

Judah L. Schwartz

I. Parametric Equations
J. Root Finder
L. Integration
N. Antiderivatives and Direction Fields
P. Conic Sections
R. Taylor Series
V. Scalar Fields
W. Vector Fields
Z. Damped Oscillator
Control-Z. Forced Damped Oscillator

Carroll O. Wilde

K. Picard's Fixed Point Method
M. Integral Evaluator
S. Complex Number Calculator
T. 3D Grapher
U. Double Integral Evaluator
X. First Order Initial Value Problem
Y. Second Order Initial Value Problem

The text of this manual was keyboarded by Maureen Emberley.

Preface

"The purpose of computation is insight, not numbers."
Richard W. Hamming

We produced The Calculus Toolkit to make it easier for people to learn and use calculus. There are programs for insight, exploration, and explanation, programs for solving equations and evaluating formulas, and programs for drill and practice. There are programs for graphing functions, generating tables of function values, investigating vector fields, graphing solutions of differential equations, evaluating definite integrals, and exploring limits, conic sections, and complex numbers. The programs work equally well in the classroom or laboratory and lend themselves to individual study as well as professional use.

Neither the programs nor this book require you to have previous computer experience or any knowledge of programming. Each program is controlled from menus that require only a few keystrokes to activate. You need only type in your numbers and formulas and push a button, so to speak, to make the programs go. All programs except the drill and practice programs allow you to enter any function you want (within reason). Every program uses the computer's graphics capabilities to the greatest extent possible and contains machine language subroutines for fast execution.

Each program is supported by a chapter in this book. The chapter begins with a statement of the program's purpose and a description of the program's general operation. It then

discusses the menus and leads you step-by-step through practice examples accompanied by reproductions of graphics displays. The chapter then concludes with exercises that guide further exploration of the program's capabilities or calculus subject.

The appendices contain information about what notation to use in entering algebraic expressions and function formulas into the computer. There is also a brief answer section.

The machine language plotting subroutines in twelve of the programs were constructed by George Lewis while he was an undergraduate mathematics major at the College of the Virgin Islands. We would like to thank him for this valuable contribution. We would also like to thank the Concourse Program students at MIT who produced early versions of some of the other programs.

We give special thanks to Richard W. Hamming of the U.S. Naval Postgraduate School for the advice and kind attention that helped to shape a number of these programs.

Many valuable suggestions came from people who reviewed the programs as they developed. We would particularly like to mention

David R. Hill, Temple University
Leslie Hogben, Iowa State University
David Isles, Tufts University
James D. Lang, Valencia Community College
Chris R. Siragusa, Cypress College
Brian J. Winkel, Rose-Hulman Institute of Technology

It is a pleasure to acknowledge the assistance in editing, review, and design that the staff of Addison-Wesley Publishing Company has given to the preparation of this work.

Any errors that may appear are the responsibility of the authors. We will appreciate having these brought to our attention.

June, 1984

R. L. F.
D. T. H.
J. L. S.
C. O. W.

A. Super * Grapher (2-D)

1. PURPOSE AND DESCRIPTION

This program graphs functions in rectangular and polar coordinates. Cartesian and polar functions are entered in the forms Y = F(X) and R = R(T). Parametric equations are entered as pairs, in the form X = X(T), Y = Y(T). Any number and combination of these functions can appear in a single display.

The program checks every formula entered for syntax before graphing. As it graphs a function, the program ignores input values that lie outside the function's domain. If you ask for a graph of SQR(X), the square root of X, for X-values from -10 to 10, for example, the program will respond by graphing the function over the interval $0 \leq X \leq 10$.

Formulas entered for functions may contain one or two arbitrary constants, C1 and C2, to which you may assign values to graph selected members from a family of curves.

You may specify the portion of the plane to be displayed on the screen, the domain of the independent variable X for cartesian graphs, the domain of the independent variable T for polar and parametric graphs, the colors of the graphs, and the number of points to be plotted and connected in making each graph (3 to 999).

Any display you generate may be saved on a spare disk for later recall.

2. STEP BY STEP

Load the program from the disk menu (allow 50 seconds), read the greeting message, and press |RETURN| to display the graph type menu shown in Screen 1. Read the menu and proceed to Example 1.

```
              |SUPER*GRAPHER|

|1|  Graph Y=F(X)    (rectangular coord.)
|2|  Graph R=R(T)   (polar coordinates)
|3|  Graph X(T) Y(T)   (parametric eq.)
|Q|  QUIT — leave program

   Press the key of your option choice.

                    _
```

Screen 1. The graph type menu.

Example 1. F(X) = SIN(X).

Starting from the graph type menu (Screen 1), press |1| to call up the cartesian formula display (Screen 2).

```
           |<CURRENT FUNCTION>|

F(X) = C1 + C2*SIN(X)

Press |RETURN| to keep this function or
type a new function and press |RETURN|.
```

Screen 2. The cartesian formula display.

The program's default (starting) formula for this display, C1 + C2*SIN(X), is the formula for a two-parameter family of functions. Press |RETURN| to accept the formula and call up a menu for setting the values of C1 and C2. The display will change to the one shown in Screen 3.

```
                    |< SETTINGS >|

     F(X) = C1 + C2*SIN(X)

     PARAMETERS:  C1 = 0
                  C2 = 1
     WINDOW:  XMIN = -3.2
              XMAX = 6.3
              YMIN = -4
              YMAX = 6

     Number of steps (3 to 999) = 120
     Color number of graph (0 to 7) = 3
     To keep a value — press |RETURN|.
     To change a value — type a new value
                          and press |RETURN|.
```

Screen 3. The cartesian settings display.

The settings display in Screen 3 shows the current values of C1 and C2, with the cursor blinking beneath the current value of C1. Unless you elect to change these values, the function that will be graphed is

F(X) = 0 + 1*SIN(X),

or simply

F(X) = SIN(X).

The settings display also shows the current X- and Y-values of the boundaries of the graphing "window," that is, the rectangular region in the plane that will appear on the screen when the function is graphed. In this case the graphing window is the region

$$-3.2 \leq X \leq 6.3, \quad -4 \leq Y \leq 6.$$

The line

Number of steps (3 to 999) = 120

that appears on the screen just below the window settings tells you that the computer will plot and connect 120 points to make the graph of F(X) = SIN(X). The step number controls the balance between the accuracy of the graph and the speed with which it is drawn. If the number of points (steps) is small, for example 3 or 10, the graph will be produced quickly, but with curved portions crudely drawn. If the number of points is large, say 200 or more, the graph will be precise but it will take a relatively long time to appear because of the number of computations involved.

You may choose one of six colors for the graph:

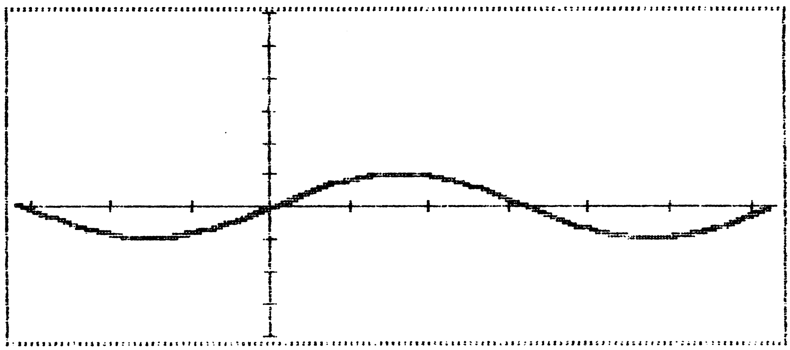
0 = black	1 = green	2 = purple	3 = white
4 = black	5 = orange	6 = blue	7 = white

The colors will vary from monitor to monitor, especially on those without tint controls. For monochrome monitors you may as well leave the color set at 3.

A black graph does not show against a black background, but it is sometimes useful to erase a graph by regraphing it in black.

To continue with the example, accept each of the current settings, one at a time, by pressing |RETURN|. With each |RETURN|, the cursor will move to the next line, leaving the displayed value in place. You will need to press eight |RETURN|s in all. After the last one, which accepts the current color value (3 for white), the menu will disappear,

and you may watch the graph of F(X) = SIN(X), $-3.2 \leq X \leq 6.3$, develop in its place. The completed graph is shown in Screen 4.



X = -3.2 TO 6.3
Y = -4 TO 6

F(X) = C1 + C2*SIN(X)
C1=0 C2=1
Press any key to see options

Screen 4. The graph of F(X) = SIN(X), $-PI \leq X \leq 2*PI$.

The display in Screen 4 shows the bounds on the plotting window, which are X = -3.2 to X = 6.3 and Y = -4 to Y = 6. The X-axis is marked in integer steps from X = -3 to X = 6. The Y-axis is marked in integer steps from Y = -4 to Y = 6. The formula F(X) = C1 + C2*SIN(X) appears toward the bottom of the display along with the parameter values C1 = 0 and C2 = 1 that determine the particular function graphed here, F(X) = SIN(X).

When you have studied the display in Screen 4, press the space bar (or any other standard key) to display a list of the options now available. The list is shown in Screen 5.

RECTANGULAR F(X)

|<OPTIONS>|

|0| See current display

|1| Same F(X) — new window
|2| New F(X) — same window
|3| New F(X) — new window
|4| Same F/window — new parameters

|C| Change graph type (Rect/Polar/Para)
|D| Save to disk

|Q| QUIT — leave program

Press the key of your option choice.

Screen 5. The rectangular graphing options menu.

Pressing

|0| Recalls the graph you just left.

|1| Keeps the current function and current values for C1 and C2, but allows you to change XMIN, XMAX, YMIN, YMAX, the number of plotted points, and the color of the graph. The graphics screen is erased, and the new graph plotted.

|2| Allows you to add a new graph to the current display. You define the new function, and enter the number of plotted points and color of its graph.

|3| Starts all over with a new function, window, number of plotted points, and color.

|4| Keeps the same function family (containing C1 and/or C2) and display but allows you to change the values of

C1 and/or C2. If your function did not contain C1 or C2, Option 4 is not shown.

|C| Sends you back to Screen 1 to change the graph type.

|D| Allows you to save your graph on a separate disk. (See Section 3, later in this chapter.)

|Q| Returns you to the disk menu.

Just for practice, press |O| to return to the graphics screen, and then |O| again to recall the options menu.

Now press |C| to return to the graph type menu (Screen 1). This will set the stage for Example 2.

Example 2. The limaçon R(T) = 1.5 + 3*COS(T).

Starting from the graph type menu (Screen 1) press |2| to call up the polar formula display, shown in Screen 6. The program's default formula for this display is

R(T) = C1*(1 + C2*COS(T)).

the formula for a two-parameter family of polar curves called limaçons. Limaçon, pronounced "LEEmasahn," is an old French word for snail.

```
|<CURRENT FUNCTION>|

R(T) = C1*(1 + C2*COS(T))

Press |RETURN| to keep this function or
type a new function and press |RETURN|.
```

Screen 6. The polar formula display.

Press |RETURN| to accept the formula and call up a menu for setting the values of C1 and C2. The display will change to the one shown in Screen 7.

|< SETTINGS >|

R(T) = C1*(1 + COS(T))

PARAMETERS: C1 = 1.5
C2 = 3
WINDOW: XMIN = -3.2
XMAX = 6.3
YMIN = -4
YMAX = 6

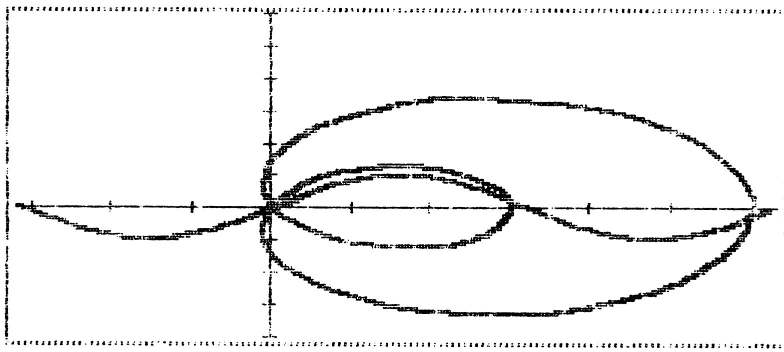
DOMAIN FOR T: TMIN = 0
TMAX = 6.3

Number of steps (3 to 999) = 120
Color number of graph (0 to 7) = 3
To keep a value — press |RETURN|.
To change a value — type a new value
and press |RETURN|.

Screen 7. The polar settings display.

The polar settings display is the same as the cartesian display except for the function formula and the lines that define the domain of T.

Press |RETURN|s to accept the current values of C1 and C2, the current bounds on X, Y, and T, and the current number of plotting steps. Then press |5| |RETURN| to graph the limaçon in orange. The graphics screen from Example 1 will reappear, and the limaçon added to it. When the plot is complete the display should look like the one shown in Screen 8.



X = -3.2 TO 6.3
Y = -4 TO 6
T = 0 TO 6.3

R(T) = C1*(1 + C2*COS(T))

C1=1.5 C2=3

|Press any key to see options|

Screen 8. The graphs of Y = SIN(X) and
R(T) = 1.5 + 4.5 COS(T), $0 \leq T \leq 6.3$, together.

After studying the graph of R(T) = 1.5*(1 + 3*COS(T)) = 1.5 + 4.5 COS(T) in relation to the graph of Y = SIN(X), press a key to call up the polar graphing options menu.

The polar graphing options menu is just like the rectangular graphing options menu shown in Screen 5, except that the top line now reads POLAR R(T) instead of RECTANGULAR F(X) and the symbols R(T) and R replace F(X) and F elsewhere in the menu.

After reading through the menu, press |0| to recall the graphics display, and |0| again to return to the menu. Then press |C| to set the stage for Example 3 by calling up the graph type menu (Screen 1).

Example 3. X(T) = 3*SIN(2*T),
Y(T) = 3*COS(3*T), 0 ≤ T ≤ 6.3.

Starting from the graph type menu (Screen 1), press |3| to call up the program's default formula for X(T), which is

X(T) = 3*SIN(C1*T).

Accept this formula by pressing |RETURN|. The program's default formula for Y(T), namely

Y(T) = 3*COS(C2*T),

will then be added to the screen. Press |RETURN| to accept this formula as well. The parametric equation settings menu will then appear on the screen. Except for the defining equations for X(T) and Y(T) at the top, the value of C1 (now 2 instead of 1.5), and the color number (changed to 5 from 3 during the course of Example 2), the menu and settings will be identical with the one shown in Screen 7.

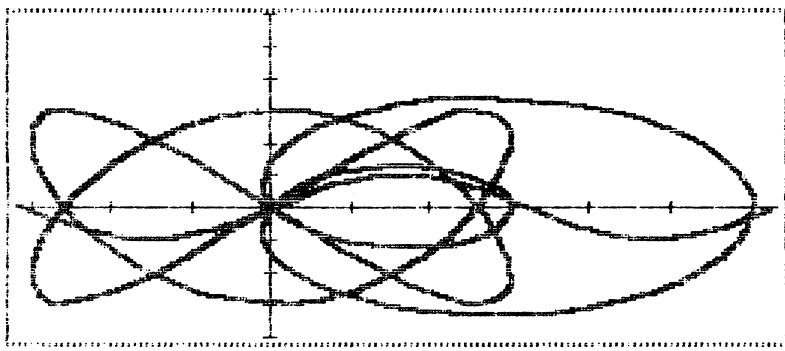
After reading through the menu, press |RETURN|s, to accept the current values for C1 and C2, the current bounds on X, Y, and T, and the current number of graphing steps. Then press |1| |RETURN| to graph the parametric equations in green. The graphics screen from Example 2, Screen 8, will reappear, and the graph of the equations

X(T) = 3*SIN(2*T), Y(T) = 3*COS(3*T), 0 ≤ T ≤ 6.3

will be added to it. The final result should agree with the one shown in Screen 9.

After viewing the three graphs, press a key to call up the parametric graphing options screen. The screen will look like the options screens for cartesian and polar equations except for the different function names.

This concludes Example 3. If you wish to practice saving a graph on a separate disk and recalling it again, then move on to Example 4, the example of the next section. If you wish instead to practice entering functions to graph, move on to Examples 5 and 6, the examples of Section 4.



X = -3.2 TO 6.3
Y = -4 TO 6
T = 0 TO 6.3

X(T) = 3*SIN(C1*T)
Y(T) = 3*COS(C2*T)

C1=2 C2=3

|Press any key to see options|

Screen 9. The graph of X(T) = 3*SIN(2*T), Y(T) = 3*COS(3*T), 0 $\le$ T $\le$ 6.3 has now been added to the display in Screen 8.

3. SAVING GRAPHS

This section deals with saving graphs for later use. You save them on a separate formatted disk, not on the Toolkit disk. While you can use the program SUPER*GRAPHER to generate and save a graph, you cannot use SUPER*GRAPHER to recall it later on. Instead, you must boot the disk that contains the recorded graph and recall the graph with the commands

HGR |RETURN|
BLOAD < FILENAME >

as described in the following example.

Example 4. Saving a graph on a separate disk.

Remove the Toolkit disk from your computer's disk drive, and put in the formatted disk on which you wish to save your graph. If you have just completed Example 3 in the previous section, the graph you save will be the one shown in Screen 9.

In any case, we suppose here that you have just generated a graph and pressed a key to call up the appropriate graph options menu. The menu will have an Option D that looks like this:

|D| Save display to disk.

Press |D| to call up the save graph menu, shown here in Screen 10.

```
                 <SAVE A GRAPH>
              SLOT = 6
             DRIVE = 1
        GRAPH NAME = SG.3*SIN(C1*T)

     PRESS |S|AVE USING THESE SETTINGS
           |C|HANGE THESE SETTINGS
           |L|IST FILES ON DISK
           |R|ETURN TO DISPLAY
           |Q|UIT
```

Screen 10. The save menu.

Normally, the disk controller (the circuit board that controls the disk drives) is in Slot 6 inside the computer, and Drive 1 is the drive you would use for SUPER*GRAPHER. If that is the case, there is no need to change the current settings.

If you have just completed Example 3, the graph name in the save display will be the one in Screen 10: SG.3*SIN(C1*T), a name composed of SG. (for SUPER*GRAPHER)

and the formula 3*SIN(C1*T) most recently used for X(T). You may save your graph under this name or whatever the current name is by pressing |S|. For practice, however, try changing the name to SIN(X).

First, press |L| to review the file names already in use on the disk on which you plan to save your graph. If the name SIN(X) appears in the catalog, use another name. (In this discussion we shall continue to use SIN(X).) After reading the name list, press |RETURN| to recall the save menu (Screen 10).

Press |C|, change the slot and drive numbers or accept them with |RETURN|s, as appropriate, and when the cursor arrives at the graph name enter SIN(X) by pressing |S| |I| |N| |(| |X| |)| |RETURN|. Then press |S| to save the graph. The word

<SAVING>

will appear on the screen while the graph is being recorded. When the process is complete, the graph itself will reappear on the screen. At this point, pressing any standard key will recall the graphing options menu and you may continue with SUPER*GRAPHER.

Instead of continuing with SUPER*GRAPHER in this example, however, let us recall the graph SIN(X) from the disk onto which it was just saved.

Press |CTRL|RESET| (together) and then press |P| |R| |#| |6| |RETURN| to boot the disk that contains the file SIN(X). (This assumes you are using Slot 6.) When the file is loaded, press

|H| |G| |R| |RETURN|

(the display screen will go blank) and then press

|B| |L| |O| |A| |D| |SPACE| |S| |I| |N| |(| |X| |)| |RETURN|

The graphs in Screen 9, or whatever graph or graphs you saved under the name SIN(X), should now appear on the screen. This concludes Example 4.

If you wish to continue with SUPER*GRAPHER, you must now return the Toolkit disk to Drive 1, and begin afresh.

4. GRAPHING YOUR OWN FUNCTION

This section gives practice with entering function formulas. The examples below involve cartesian curves, but the methods illustrated apply to polar and parametric curves as well.

Example 5. F(X) = COS(X + 1), -2*PI ≤ X ≤ PI.

Starting from the graph type menu shown in Screen 1, press |1| to call up the cartesian formula display (Screen 2).

If you wanted to graph the displayed equation, F(X) = C1 + C2*SIN(X), you would just press |RETURN|. To graph a different equation, just type the new equation over the old one and press |RETURN|. In this case, the new equation is shorter than the old one, but don't worry about the characters that remain on the screen from the old equation; they will disappear when you press |RETURN| at the end of the new equation.

If you make a typing mistake, use the left arrow key, |<-|, to go back and overstrike the erroneous character with the correct one. Then use the right arrow key, |->|, to return to the end of the formula and press |RETURN|.

The computer will check every formula you enter for correct syntax to be sure that it makes sense mathematically. To see what happens if you leave out the last parenthesis when you attempt to enter F(X) = COS(X + 1), press

|C| |O| |S| |(| |X| |+| |1| |RETURN|.

The computer will beep and display the message

Your function is not properly defined.
Please try again.

The cursor will then return to the first character of the formula you keyed in, in this case "C," and wait for you to

enter a correct formula.

To continue the example, press the right arrow key, |->|, seven times to move the cursor to the place where the missing right parenthesis should go:

F(X) = COS(X + 1_

Type a right parenthesis, |)|, followed by |RETURN|.

You should now be ready to set the boundaries of the graphing window. Press

|-| |2| |*| |P| |I| |RETURN|

to enter -2*PI (Toolkit notation for -2π) as the new XMIN. Then enter π as the new XMAX by pressing

|P| |I| |RETURN|.

Accept the displayed values

YMIN = -4, YMAX = 6

by pressing two more |RETURN|s.

Next, enter 20 for the number of steps by pressing

|2| |0| |RETURN|

and select green for the graph color by pressing |1| |RETURN|. The computer will now graph F(X) = COS(X + 1) over the interval -2*PI ≤ X ≤ PI.

After examining the graph, notice that the data on the screen give the eight-place decimal approximations to -2*PI and PI.

Now press a key to move on to the graphing options menu for the next example.

Example 6. Graph the functions F(X) = COS(2*X + PI/3) and G(X) = COS(-.5*X - 1) in a common graph over the interval -2*PI ≤ X ≤ 4*PI.

Method 1. Graph F and add the graph of G to the display.

Starting from the cartesian graphing options menu (Screen 5) press |3| for "New F(X)—new window." When the

cartesian formula display appears, press

|C| |O| |S| |(| |2| |*| |X| |+| |P| |I| |/| |3| |)| |RETURN|

to enter the formula for F(X) = COS(2*X + PI/3). Then change the window settings as necessary to have

XMIN = -2*PI XMAX = 4*PI YMIN = -4 YMAX = 6

Set the number of steps at 120, and the color number at 3 (white). The return you press to enter the color number will start the graphing.

After studying the graph, press a key to return to the graphing options menu.

To add the graph of COS(-.5*X - 1) to the graph you have just drawn, you must use the same graphing window. Therefore, press |2| for "New F(X)—same window," and then press

|C| |O| |S| |(| |-| |.| |5| |*| |X| |-| |1| |)| |RETURN|

to enter the new formula. Press

|4| |0| |RETURN|

to set the number of graphing steps at 40 (just for practice), and then press |1| |RETURN| to request the graph in green. The graphs in the final display should look like the ones in Screen 11.

Now press a key to return to the graphing options menu and prepare for Method 2.

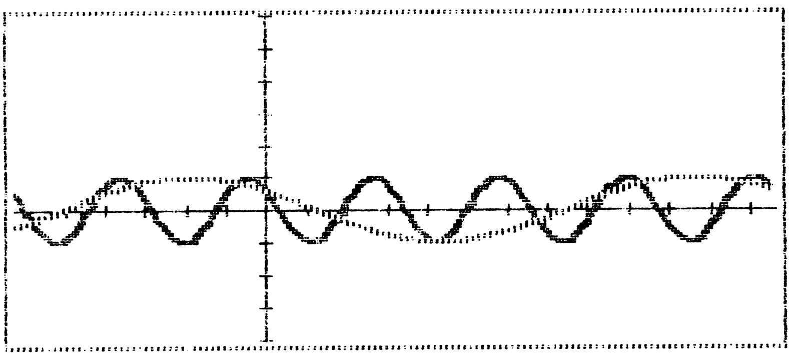
Method 2. In this method we graph the functions COS(2*X + PI/3) and COS(-.5*X - 1) together as members of the family

F(X) = COS(C1*X + C2).

To erase the old graphs and start with a clean graphics display, press |3| on the graphing options menu for "New F(X)—new window." When the cartesian formula display appears, press

|C| |O| |S| |(| |C| |1| |*| |X| |+| |C| |2| |)| |RETURN|

to enter the formula F(X) = COS(C1*X + C2).



X = -6.28318531 TO 12.5663706
Y = -4 TO 6

F(X) = COS(-.5*X - 1)

|Press any key to see options|

Screen 11. Graphs of COS(2*X + PI/3) and COS(-.5*X - 1).

Next, enter the values

C1 = 2   C2 = PI/3

to select F(X) = COS(2*X + PI/3) as the function to be graphed.

The displayed window settings should be the same as in Method 1; accept them by pressing returns. Then set the number of plotting steps at 120, and the color number at 3 to graph the function.

After viewing the graph to be sure everything is as it should be, press a key to return to the graphing options menu.

Now press |4| and enter the values

C1 = -.5   C2 = -1

that select F(X) = COS(-.5*X - 1) as the function to be graphed. Then set the number of plotting steps at 40 (for faster drawing) and the color number at 1. The final display should be just like the one in Screen 11, except that the bottom lines now read

F(X) = COS(C1*X + C2)
C1 = -.5 C2 = -1.

This concludes Example 6.

PROBLEMS

Use SUPER*GRAPHER to graph each of the following functions.

Rectangular coordinates, Y = F(X)

1. F(X) = SQR(1 - X*X), $-2 \leq X \leq 2$, $-1 \leq Y \leq 2$.
   Use 200 steps. Because of the way computer graphics displays are organized, the appearance of some graphs may be distorted. For example, the graph of F(X) = SQR(1 - X*X) may be elliptical instead of circular. The distortion can be minimized by choosing window dimensions that satisfy the equation

   (XMAX - XMIN) = 1.6*(YMAX - YMIN).

   To see the difference this makes, regraph F(X) = SQR(1 - X*X) with XMIN = -2.4 and XMAX = 2.4.
2. F(X) = C1*SQR(1 - X*X) with C1 = 1 and C1 = -1 in a common display.
3. F(X) = 1/SQR(3 - X), $-1 \leq X \leq 5$, $-2 \leq Y \leq 7$, and the number of plotted points set at 200.
4. F(X) = INT(X), $-5 \leq X \leq 5$, $-5 \leq Y \leq 5$. (INT(X) is the greatest integer that is less than or equal to X.)
5. F(X) = X/SQR(X*X + 1), $-10 \leq X \leq 10$, $-1 \leq Y \leq 1$.
6. F(X) = (X*X + 1)/(X^3 - 4*X), $-5 \leq X \leq 6$, $-4 \leq Y \leq 5$.
   This function has three vertical asymptotes, so set the number of plotted points at 240.
7. F(X) = X + 1/X

8. F(X) = X*X - X + 1
9. F(X) = (X^3)/3 - X*X/2 - 2*X + 1/3
10. F(X) = X^3 - 27*X + 36
11. F(X) = X/(X + 1)
12. F(X) = SIN(1/X), $-2 \leq Y \leq 2$, for
    a) $-5 \leq X \leq 5$, b) $-1 \leq X \leq 2$, c) $-.5 \leq X \leq .3$
13. F(X) = X*SIN(1/X), for
    a) $-1 \leq X \leq 1$, $-1 \leq Y \leq 1$
    b) $-.2 \leq X \leq .3$, $-.3 \leq Y \leq .25$
    c) $-.02 \leq X \leq .03$, $-.015 \leq Y \leq .015$
    d) $-2*PI \leq X \leq 3*PI$, $-.4 \leq Y \leq 1.2$
14. F(X) = SIN(X) + .7*(RND(1) - .5)

Polar coordinates, R = R(T)

15. R(T) = SIN(C1*T), taking C1 = 1, 2, 3, 4, . . . What effect does multiplying the variable T by a number have? (Take X and Y from -1.2 to 1.2.)
16. R(T) = SIN(T + C1), taking C1 = 0, .5, 1, 1.5, . . . What effect does adding a constant to the variable T have? (Take X and Y from -1.2 to 1.2.)
17. R(T) = C1 + SIN(T), taking C1 = 0, 1, 2, . . . What effect does adding a constant to the function R(T) have? (Take X and Y from -1.2 to 3.5.)
18. R(T) = SIN(C1*T)*COS(C2*T), R(T) = SIN(C1*T)*SIN(C2*T), and R(T) = SIN(C1*T). Different choices of C1 and C2 will produce "stars," "flowers," and "butterflies."
19. R(T) = 1 + SIN(2*T)*COS(3*T), $0 \leq T \leq 2*PI$.
20. R(T) = 10/T, taking T from -4 to 12, X from -8 to 20, and Y from -8 to 8. The curve is a hyperbolic spiral.
21. R(T) = C1*COS(T). Use different colors to graph R(T) with C1 = +2, -2, +4 and -4, take T from 0 to 3.2, X from -1 to 5, and Y from -5 to 5. How do the magnitude and sign of C1 affect the graph?
22. R(T) = C1/(1 - C1*COS(T)). This is the standard polar equation for the conic sections. Try C1 = .5, .8, 1, 1.4, 2. For positive values of C1 less than 1, what conic section do you get? For C1 = 1, what conic

section do you get? What conic section do you get by taking C1 greater than 1?

Parametric Equations X = X(T), Y = Y(T)

23. Trochoids: X(T) = C1*T - C2*SIN(T)
    Y(T) = C1 - C2*COS(T)

    Each of these curves can be thought of as the path of a point (attached C2 units from the center) on a wheel (of radius C1) that is rolling along the X-axis. If C1 = C2, then the point is on the edge of the wheel, and the resulting graph (and path) is called a cycloid. If C2 is greater than C1, then the point can be thought of as being on a flange that extends beyond the edge of the wheel, and the resulting path contains loops that reflect retrograde motion.
24. Epicycloids:
    X(T) = C1*COS(T) - C2*COS(T*C1/C2)
    Y(T) = C1*SIN(T) - C2*SIN(T*C1/C2)

    Each of these curves can be thought of as the path of a point on a wheel that is rolling around the outside edge of another wheel. Nice graphs result when C1 is greater than C2 and C2 divides C1.
25. Hypocycloids: X(T) = C1*(COS(T))^3
    Y(T) = C1*(SIN(T))^3
26. Involutes of a circle:
    X(T) = C1*COS(T) + C1*T*SIN(T)
    Y(T) = C1*SIN(T) - C1*T*COS(T)
27. The Witch of Maria Agnesi:
    X(T) = C1*COS(T)/SIN(T)
    Y(T) = C1*SIN(T)*SIN(T)

5. FINDING ROOTS AND POINTS OF INTERSECTION GRAPHICALLY

By inspecting the graph of a function you can sometimes estimate its roots (points where the graph crosses or touches the X-axis) or the coordinates of the points where

it meets another curve. When you wish to use a numerical root finder or equation solver, it can help to know beforehand how a function behaves.

To learn more about the location of a root or intersection point, you can use SUPER*GRAPHER as a "magnifying glass" to expand a small area of the plane to fill the entire graphical display.

There is a way to save time and typing when you are trying to find the points of intersection of the graphs of two functions, say A(X) and B(X). To avoid retyping the formulas for A and B every time you change windows, enter F(X) = C1*A(X) + (1 - C1)*B(X) and then assign C1 the values 0 and 1 after each window change. Putting C1 = 0 will set F(X) = B(X), and putting C1 = 1 will set F(X) = A(X). Thus, A(X) and B(X) can be graphed quickly by changing the coefficient C1.

PROBLEMS

SUPER*GRAPHER can sometimes be used to approximate the limit of a function F(X) as X approaches a value C by graphing the function in a neighborhood of C. Use the "magnifying glass" idea to estimate the following limits.

1. Limit SIN(X)/X
   X -> 0
2. Limit SIN(2*X)/(3*X)
   X -> 0
3. Limit (1 - COS(X))/(X + X*X)
   X -> 0
4. Limit X^(1/X)
   X -> 0+

SUPER*GRAPHER can sometimes be used to estimate limits as X approaches infinity by examining the graph of F(X) for very large values of X. Estimate the following limits by graphing each function for large values of X.

5. Limit (2*X*X + 5*X + 34)/(5*X*X - 734)
   X -> ∞
6. Limit (1 + 1/X)^X
   X -> ∞
7. Limit SQR(X + 54) - SQR(X)
   X -> ∞
8. Limit ATN(X)
   X -> ∞

6. FINDING MAXIMA AND MINIMA

The techniques of calculus are useful for finding extreme values (maxima and minima) of functions, and SUPER*GRAPHER can help check your results. It can also provide quick preliminary estimates of extreme values.

PROBLEMS

1. **Autocatalytic reactions:** A catalyst for a chemical reaction is a substance that controls the rate of the reaction without undergoing any permanent change in itself. An autocatalytic reaction is one whose product is a catalyst for its own formation. Such a reaction may proceed slowly at first if the amount of catalyst present is small, and slowly again at the end when most of the original substance us used up. But in between, when both the substance and its product are relatively abundant, the reaction may proceed at a faster rate. In some cases it is reasonable to assume that the rate V = DX/DT of the reaction is proportional both to the amount X of product and to the amount of the original substance still present. That is, V may be considered to be a function of X alone, and V = C1*X*(C2 - X), where C2 is the amount of substance at the beginning, and C1 is a positive constant.

   a) Graph V for C1 = 1 and try C2 = 4, 6, 8. Fix C1 = 2 and try C2 = 4, 6, 8. How does the location of the maximum seem to depend on the values of C1 and C2?

b) At what value of X does V have a maximum on the closed interval from 0 to C2? What is the maximum value of V?

2. **Square box, no top:** A C1 by C1 square piece of material is to be made into a box with a square base by cutting an X by X piece from each corner and folding up the sides. The volume of the box is F(X) = X*(C1 - 2*X)^2, and X lies between 0 and (C1)/2. Estimate the value of X that maximizes the value of F. Try C1 = 4, 10, and 20. Where are the maxima for these values of C1? How does the location of the maximum depend on C1?

3. **Snell's Law:** According to Fermat's principle in optics, light follows the path of least travel time. Suppose you are interested in the path that a ray of light traveling in the XY-plane will take in going from the point P(0,1) to the point Q(2,-1) if the velocity of light has the constant value C1 in the medium above the X-axis and the constant value C2 in the medium below the X-axis. (See Fig. 1 on the next page.) In either medium, where the velocity of light is constant, least time means least distance, and the light travels along a straight line. Hence the light's path from P to Q will consist of a line segment from P to a point R(X,0) on the boundary between the two media, followed by another line segment from R to Q.

According to the formula time = distance/rate, the travel time from P to R is SQR(1 - X*X)/C1 and the travel time from R to Q is SQR(4 - (2 - X)*(2 - X))/C2 or SQR(4*X - X*X). The total time it will take for the light to travel from P to Q is therefore

F(X) = SQR(1 - X*X)/C1 + SQR(4*X - X*X)/C2.

Graph F for various values of C1 and C2 and estimate the location of R(X,0) that minimizes F(X) in each case. You might start with C1 = 1 and C2 = 1.2. Be sure to investigate cases in which C1 = C2.

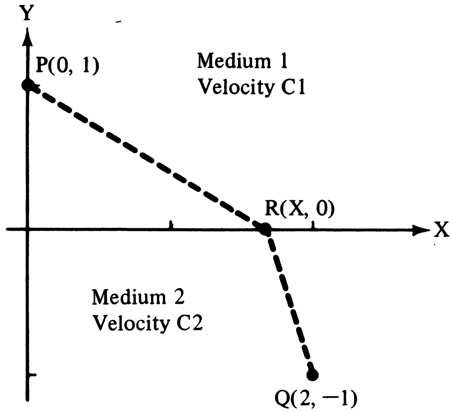


Figure 1. Diagram for Problem 4.

4. **Cough:** When you cough, your trachea (windpipe) contracts to increase the velocity of the air you expel. If C1 is the radius of the trachea at rest and X is the contracted radius, then F(X) = K*(X - C1)*X*X is a reasonable model for the velocity of the expelled air. Let K = 1 and estimate graphically the value of X that maximizes F for various values of C1.

5. Estimate the extreme values of each of the following functions on the closed interval from 0 to 2*PI.

   a. F(X) = SIN(X) + COS(X)   b. F(X) = SIN(X) - COS(X)

   c. F(X) = SIN(X)*COS(X)

B. Name That Function

1. PURPOSE

This program provides experience in working with parameters in formulas for families of curves.

2. DESCRIPTION

You select a family of linear, quadratic, cubic, sinusoidal, rational, or exponential functions, and one of three levels of difficulty. The program then randomly selects a member of the family with integer coefficients in a small range and displays the graph. You are asked to identify the particular curve shown by finding the values of the coefficients. The function corresponding to your choice is then graphed. You are then told whether or not your choice was correct, and you may try again if it was not. The correct answer is given after ten incorrect attempts, but you may ask for it earlier. A score based on the number of attempts and level of difficulty allows you to monitor your progress.

3. STEP BY STEP

Load the program from the main disk menu. After reading the greeting message, press |RETURN| to call up the menu of available functions, shown in Screen 1.

```
|<AVAILABLE FUNCTIONS>|
  1 ... LINEAR
  2 ... QUADRATIC
  3 ... CUBIC
  4 ... SINE
  5 ... EXPONENTIAL
  6 ... RATIONAL

PRESS THE NUMBER OF THE
   FAMILY YOU WANT.
```

Screen 1. The menu of available functions.

After reading the menu, press |4| to work with sine functions. A level-of-difficulty menu will appear below the list of function types, as shown in Screen 2. Press |3|

```
        |<AVAILABLE FUNCTIONS>|
           1 ... LINEAR
           2 ... QUADRATIC
           3 ... CUBIC
       ==> 4 ... SINE
           5 ... EXPONENTIAL
           6 ... RATIONAL

        |<LEVEL OF DIFFICULTY>|
           1 ... EASIEST
           2 ... MODERATE DIFFICULTY
       ==> 3 ... HARDEST

PRESS  |RETURN|  TO KEEP THESE SETTINGS
 OR      |C|     TO CHANGE THEM
 OR      |Q|     TO LEAVE PROGRAM
```

Screen 2. Having chosen to work with sines, you may choose one of three levels of difficulty. An arrow indicates your choice.

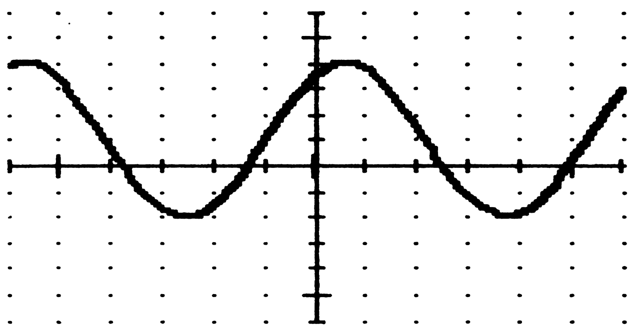
for HARDEST. An arrow will indicate your choice. You may still change your mind at this point (press |C| to start over), but to continue the demonstration, press |RETURN|.

From here on, the pictures on your screen will probably not agree with ours because the problems are selected randomly. The pictures that follow, however, show a typical run through the program.

When we pressed |RETURN| to request a problem, the computer showed us the graph in Screen 3.

|Screen 3|

PROBLEM #1 TRY #1 SCORE = 100



FAMILY: F(X) = A+B*SIN(C*X+D)

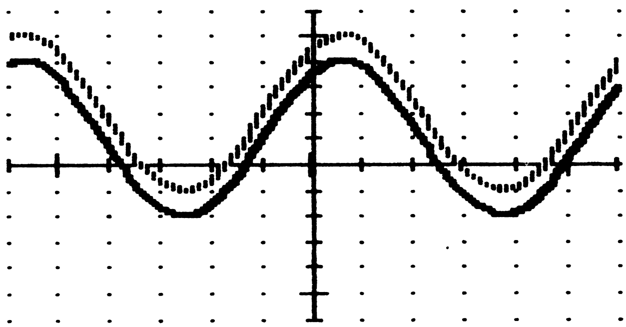
PRESS	A (-3 TO 4)	= 2
|C|HANGE VALUES	B (-3 TO 3)	= 3
|T|EST VALUES	C (-2 TO 2)	= 1
|R|EVIEW OLD TRY	D (-2 TO 2)	= 1
|S|EE ANSWER/STOP		

Our task was to choose the values of the parameters A, B, C, and D that would make the graph of the function F(X) = A + B*SIN(C*X + D) match the graph on the screen. In this program, the correct values are always integers. The default values of the parameters (the values that first appeared on the screen) were all 1's. We changed the value

of A to 2 by pressing |C| followed by |2| |RETURN|. After that, we changed the value of B to 3 by pressing |3| |RETURN|. We then pressed two more |RETURN|s to accept the default values for C and D. This produced the values and message shown in Screen 3 on the previous page.

|Screen 4

PROBLEM #1 TRY #2 SCORE = 100



FAMILY: F(X) = A+B*SIN(C*X+D)

PRESS		
|C|HANGE VALUES	A (-3 TO 4)	= -3
|T|EST VALUES	B (-3 TO 3)	= 3
|R|EVIEW OLD TRY	C (-2 TO 2)	= 1
|S|EE ANSWER/STOP	D (-2 TO 2)	= 1
	|*WRONG*|	

We then pressed |T| to test our parameter choices, and when the computer displayed the graph of

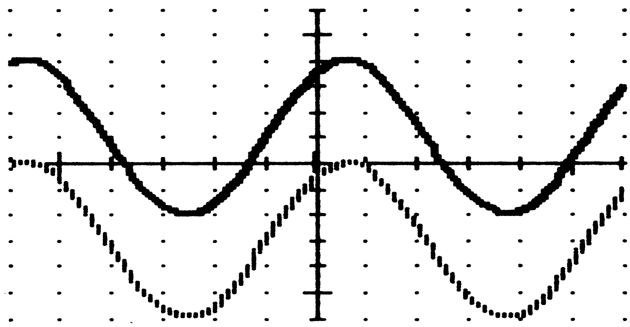
F(X) = 2 + 3*SIN(X + 1),

the function determined by our parameter choices, it also displayed the message "WRONG."

We then changed the value of A to −3 by pressing |C| |-| |3| |RETURN|, kept the displayed values of the other parameters by pressing |RETURN|s, pressed |T| again, came up wrong once more (Screen 4), and pressed |S| to see the correct parameter values for the problem, shown in Screen 5.

|Screen 5|

PROBLEM #1 TRY #3 SCORE = 94



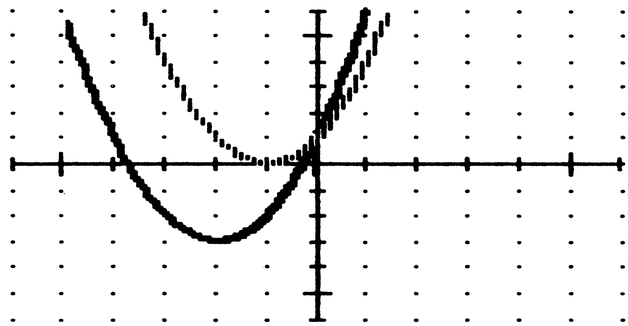
FAMILY: F(X) = A+B*SIN(C*X+D)

PRESS	A (−3 TO 4) = 1
|A|NOTHER PROBLEM	B (−3 TO 3) = 3
|M|ENU OF TYPES	C (−2 TO 2) = 1
|Q|UIT	D (−2 TO 2) = 1
	|CORRECT ANSWERS|

We then pressed |M| to return to the function menu and chose quadratic functions by pressing |2|. Screens 6, 7, and 8 show a quadratic problem we got right on the third try.

Screen 6

PROBLEM #1 TRY #1 SCORE = 100

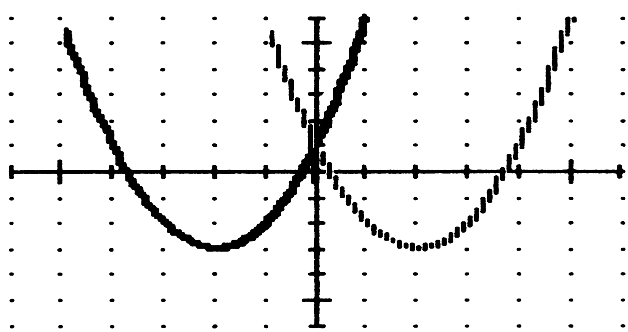


FAMILY: F(X) = A*X*X+B*X+C

PRESS RETURN TO A (-2 TO 2) = 1
KEEP A VALUE OR B (-4 TO 4) = 2
TYPE A NEW VALUE C (-4 TO 4) = 1
AND PRESS RETURN * WRONG *

Screen 7

PROBLEM #1 TRY #2 SCORE = 94

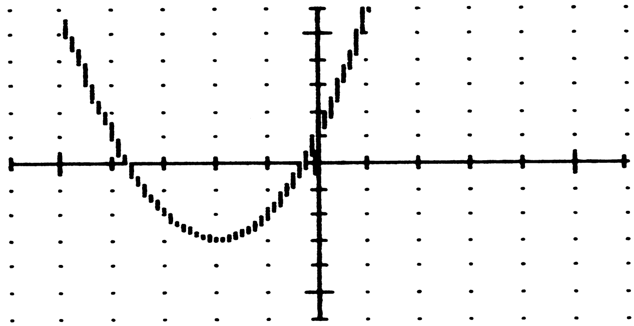


FAMILY: F(X) = A*X*X+B*X+C

PRESS RETURN TO A (-2 TO 2) = 1
KEEP A VALUE OR B (-4 TO 4) = -4
TYPE A NEW VALUE C (-4 TO 4) = 1
AND PRESS RETURN * WRONG *

|Screen 8|

PROBLEM #1 TRY #3 SCORE = 88



FAMILY: F(X) = A*X*X+B*X+C

PRESS |RETURN| TO A (-2 TO 2) = 1
KEEP A VALUE OR B (-4 TO 4) = 4
TYPE A NEW VALUE C (-4 TO 4) = 1
AND PRESS |RETURN| |* RIGHT *|

4. THE KEY-PRESS OPTIONS

Now work your way through a problem on your own. For each try, you have the following options:

|C| Lets you change the coefficients.

|T| Graphs your function and compares it with the displayed function. If the graphs match exactly, you are correct and may try another problem (press |A|), try another family (press |M|), or quit (press |Q|).

|R| Lets you see the graphs and coefficients of any earlier tries (after pressing |R|, enter the try number and press |RETURN|).

|S| Lets you see the correct answer. Pressing |S| also calls up the |A|-|M|-|Q| options.

C. Secant Lines

1. PURPOSE AND DESCRIPTION

By a visual presentation of the limiting process, this program reinforces the idea that the tangent line at a point on a curve is a limit of secant lines. As you command the program to draw secants that approach a tangent through a selected point on a function's graph, the screen also shows the slopes of the secants in a numerical display. The program accepts any of the standard functions from calculus, and offers a menu of functions like SIN(X), ABS(X*X - 4) + 3, and INT(X) for initial experimentation.

2. STEP BY STEP

Load the program from the main disk menu, read the greeting message, and press |RETURN| to display the menu of available functions, shown on the following page. The examples in this chapter all start from the function menu.

Example 1. A Differentiable Function: F(X) = SIN(X)

Press |2| on the function menu to call up the current settings menu for F(X) = SIN(X). This menu enables you to set the stage by choosing the point (X0,F(X0)) through which the secant lines will all pass and at which you wish to explore the question of whether a tangent line exists.

```
              |<AVAILABLE FUNCTIONS>|

               1... DEFINE YOUR OWN
               2... SIN(X)
               3... X*X-4
               4... X*X-4*X+4
               5... 2*(X^3)-3*(X*X)-12*X
               6... SQR(25-X*X)
               7... ABS(X)
               8... ABS(X*X-4)+3
               9... INT(X)
            PRESS THE NUMBER OF THE
               FUNCTION YOU WANT.
```

Screen 1. The function menu.

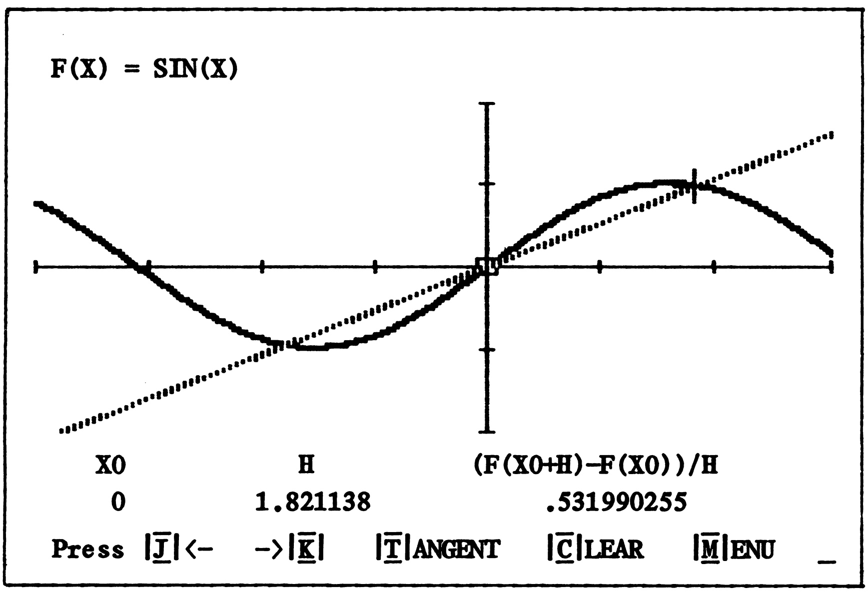
```
              |<CURRENT SETTINGS>|

     F(X) = SIN(X)

    XMIN = -4              YMIN = -2
    XMAX = 3               YMAX = 2
      X0 = 0              F(X0) = 0

  PRESS  |RETURN|  TO KEEP THESE SETTINGS
            |ESC|  TO CHANGE SETTINGS
              |F|  TO CHANGE F(X)
              |Q|  TO QUIT
```

Screen 2. The current settings menu.

The current settings menu also enables you to set the X and Y boundaries of the viewing window that will appear on the graphics screen. The program will not accept values of XO that lie outside the interval XMIN $\leq$ XO $\leq$ XMAX. If you enter an unacceptable value, the program will tell you so and give you an opportunity to redefine XO or change the graphing window.

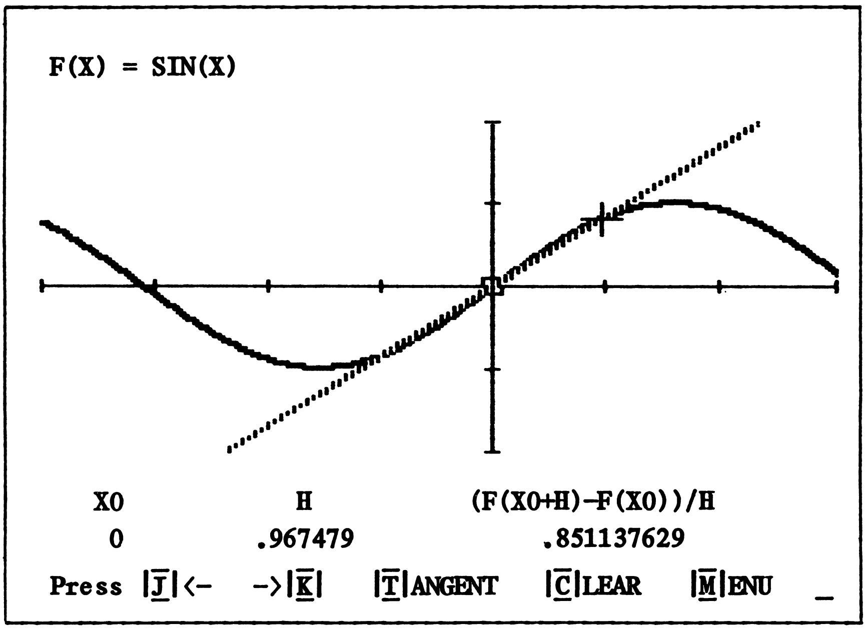
The program will warn you if you choose a value of XO at which F is not defined, and give you an opportunity either to change XO or define F(XO), as in Example 4.

Press |RETURN| to keep the current settings. The screen will go blank. Then, in order of appearance, you will see coordinate axes, the graph of F(X) = SIN(X), the secant line through the points (XO, F(XO)) = (0,0), and the point (XO,F(XO + H)). The point (0,0) will be enclosed in a small square and the point (XO,F(XO + H)) marked with a crosshair. Numerical information and operating instructions will appear at the bottom of the screen. When the display is complete, it will look like the one in Screen 3.



Screen 3. The sine curve and initial secant line.

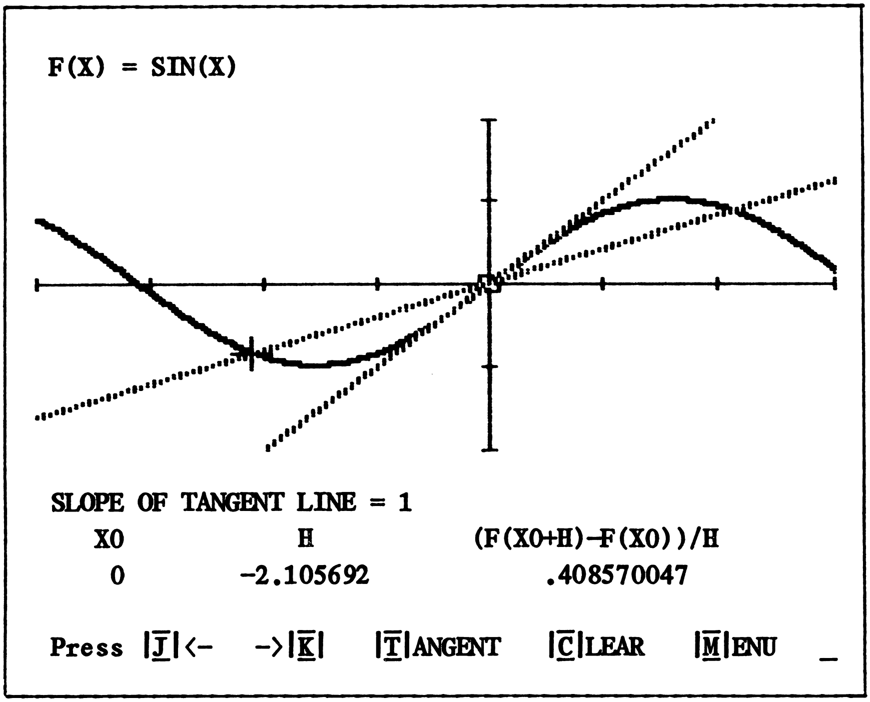
You can move the crosshair left or right along the graph in small steps by pressing |<-| or |->|, and in larger steps by pressing |J| or |K|. Try all four. The arrow keys are particularly useful for homing in on a point once you are nearby. The secant line will be redrawn after each keypress. Watch the display of the numerical values of H and the secant slope change as the crosshair moves.



Screen 4. Pressing |J| once changes the display shown in Screen 3 into the one shown here.

After you have experimented with moving the secant line, move it to the position shown in Screen 5. Then press |T| to display the tangent line.

Now move the crosshair to the origin. The secant line will disappear from the screen, although the tangent line will remain in place. Also, the word "UNDEFINED" will appear



Screen 5. The tangent line may be added to the display at any time.

in place of the numerical secant slope display. The tangent slope is defined at X0 = 0 (it equals 1) but the difference quotient (F(X0+H) - F(X0))/H is no longer defined now that H = 0.

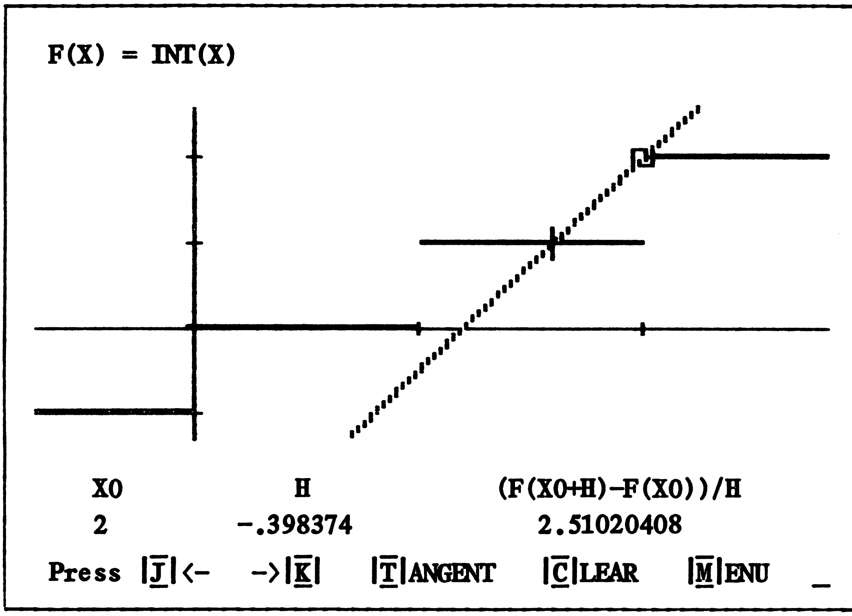
If you wish to remove the tangent line from the picture, press |C| for "CLEAR." The computer will respond by redrawing the graphics display in its original state.

Now press |M| to return to the current settings menu (Screen 2), and press |F| to return to the function menu (Screen 1) for a new function.

Example 2. A Discontinuous Function: F(X) = INT(X)

Starting out from the function menu (Screen 1), press |9| to select F(X) = INT(X), the greatest integer function.

Then press |RETURN| to accept the default values of XO and the other parameters. When the graph appears, watch the secant lines as you move the crosshair to the position shown in Screen 6.



Screen 6. A discontinuous function.

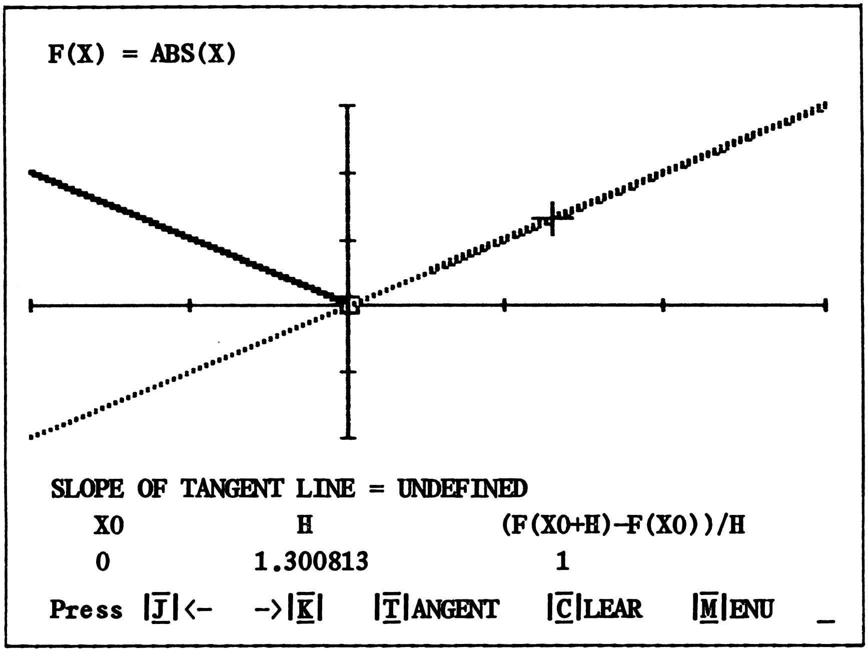
Now press |T| to call for the tangent to the graph of F(X) = INT(X) at the point (2,2). There is no tangent to the graph at the point (2,2), as indicated by the message

SLOPE OF TANGENT LINE = UNDEFINED

Return to the function menu, when you are ready, by pressing |M| and then |F|.

Example 3. A Continuous Function with a Singular Point: F(X) = ABS(X)

Starting from the function menu in Screen 1, press |7| to select the absolute value function F(X) = ABS(X). Then press |RETURN| to accept the displayed current settings and request the graph.



Screen 7. The graph of F(X) = ABS(X) has no tangent at the origin.

Experiment with secant lines through points to the right and left of (XO,|XO|) = (0,0). They are all extensions of the two "branches" of the function's graph.

Return the crosshair to its original position, shown in Screen 7, and press |T| to request the tangent line to the graph F(X) = ABS(X) through the origin. As in Example 2, the program will respond by adding the words

SLOPE OF TANGENT LINE = UNDEFINED

to the display.

Now press |M| and then |F| to return to the function menu for the next example.

Example 4. A Removable Discontinuity:

F(X) = X*SIN(1/X)

Starting from the function menu (Screen 1), press |1|. When the prompt

TYPE YOUR FUNCTION AND PRESS |RETURN|

appears, press

|X| |*| |S| |I| |N| |(| |1| |/| |X| |)| |RETURN|

to enter F(X) = X*SIN(1/X). Then press |ESC|. When

X0 = 0

is displayed, press |RETURN| to accept it. The line

F(X0) = UNDEFINED

will appear almost immediately. Press |0| |RETURN| to define F(X0) to be zero. With F(X0) so defined, the function entered into the computer is the continuous function

$$F(X) = \begin{cases} X*SIN(1/X), & X \neq 0 \\ 0, & X = 0. \end{cases}$$

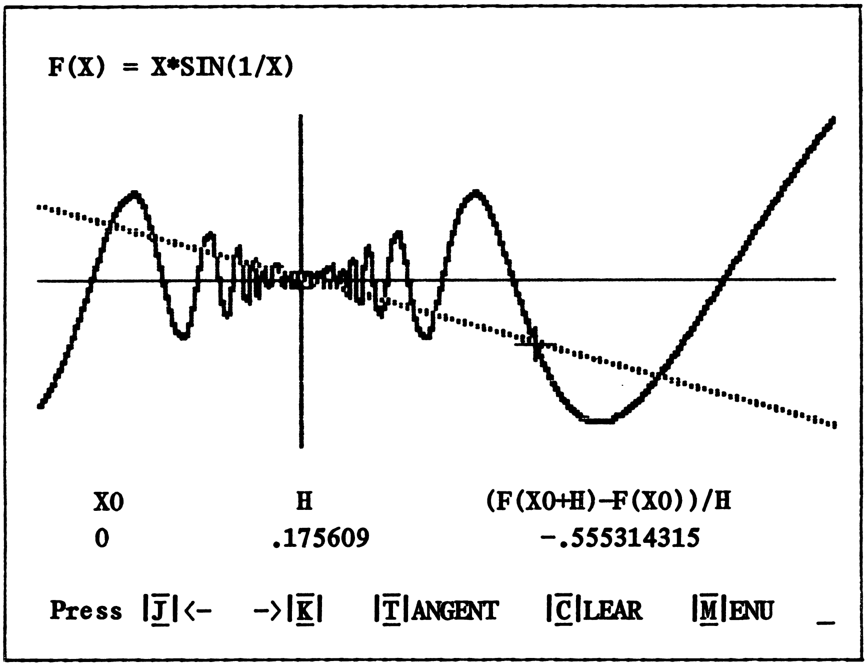
Now press |ESC| once more to enter the window parameters. The program will ask you to reconfirm your choice of values for X0 and F(X0), so begin by pressing |RETURN| to accept X0 = 0 when it appears, and then reenter F(X0) = 0 by pressing |0| |RETURN|. This clears the way for the window parameters to appear. When they do, enter

XMIN = −.2 XMAX = .4 YMIN = −.25 YMAX = .25

Finally, press |RETURN| to see the graph shown in Screen 8.

For practice, move the crosshair to the position shown in Screen 8.

Press |<-| repeatedly to move the crosshair to the origin, watching the erratic behavior of the secant line.



Screen 8. The graph of the continuous extension of F(X) = X*SIN(1/X) contains the origin.

The values displayed for the secant's slope will vary from close to -1 to nearly +1. (If the crosshair were moving continuously, the values -1 and +1 would actually be taken on.)

Next, press |T| to request the tangent line through the origin. As you might expect from the secant's behavior, the secant never settles down as H->0, and no tangent exists at the origin. The continuous extension of F(X) = X*SIN(1/X) to the origin is not differentiable there.

In contrast, the continuous extension of the function F(X) = X*X*SIN(1/X) to the origin is differentiable at X0 = 0, and Problem 9 will ask you to explore the behavior of the secants to its graph through the origin as H->0.

3. SLOPE OF TANGENT LINE = UNDEFINED(?)

The program SECANT LINES estimates the slope of the tangent line by computing the ratio (F(X0 + H) - F(X0))/H for two small values of H, namely H = -.0000002 and H = .0000002. If the two values of the ratio obtained this way are close (absolute difference less than .0001) the program reports the slope of the tangent line to be the average of the two values. For many functions this average is a useful approximation, but two things can go wrong:

i) Since the program is dividing by such small values of H, round-off and truncation can be a problem.

ii) If the values of the ratios obtained from H = -.0000002 and H = .0000002 are more than .0001 units apart, then the program reports that the slope of the tangent line may be undefined. This may look like a handy way to detect undefined derivatives, but a differentiable function with a very steep graph, say a slope of magnitude 10,000 or more at X0, will be presented as a function that may have no tangent line at X0 (as in Example 5), while a function that has acceptable ratios for these small H's will be assigned a tangent line even if it has no derivative at X0 (as in Problem 10).

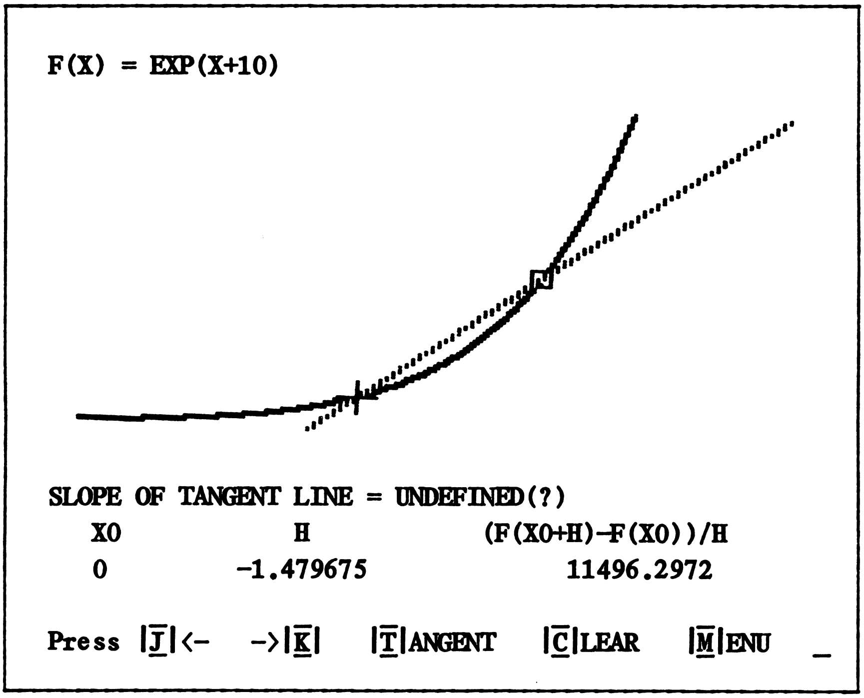
Example 5. An inconclusive report on the function F(X) = EXP(X+10).

Press |1| on the function menu and enter the formula for F(X) = EXP(X + 10). When Screen 2 comes up press |ESC| and enter the following parameter settings:

X0 = 0
XMIN = -4
XMAX = 2.5
YMIN = 500
YMAX = 45000

Press |RETURN| to display the graph. Experiment with secants for a bit, if you wish, then ask for the tangent.

Screen 9 shows what appeared after we pressed |J| four times and then pressed |T|.



Screen 9. There is a limit to how steep a slope the program can handle. When this limit is exceeded, the computer reports that the slope you requested may be undefined.

The trade-off. To enable you to enter your own functions, and not limit you to a preselected list, it was necessary for the program to contain a general secant slope estimator. Slope estimators can be fooled, but we felt that the ability to enter a variety of formulas outweighed the possible difficulties.

PROBLEMS

In Problems 1-3, use SECANT LINES to observe the convergence of the secant lines to the tangent lines. For each function, build a table of values of the slopes of the secant lines as the crosshair moves along the graph of F(X) toward the point (X0,F(X0)), i.e., as H approaches 0.

1. F(X) = SIN(X) (#2 on the function menu)
2. F(X) = X*X - 4 (#3 on the function menu)
3. F(X) = 2*X 3 - 3*X*X - 12*X (#4 on the function menu)

In Problems 4-8, use SECANT LINES to estimate the slope of the line tangent to the graph of F(X) at the point (X0,F(X0)) for the indicated values of X0.

4. F(X) = X*X. Take X0 = 0, 1, 2, 3, -1, -2, 0.5, 0.7, and 1.5. What is the pattern?
5. Put F(X) = X^3. Take X0 = 0, 1, 2, 3, -1, -2, -3, 0.5, and 0.7. What is the pattern?
6. F(X) = LOG(X). Take X0 = 1, 1.5, 2, 3, and 5.
7. F(X) = EXP(X). Take X0 = 0, 1, and 2.
8. F(X) = SIN(X). Take X0 = 0, PI/4, PI/2, PI, and 2*PI.

9. Repeat the steps of Example 4 with the function F(X) = X*X*SIN(1/X). Use X0 = 0, F(X0) = 0, XMIN = -.07, XMAX = .12, YMIN = -9E-03, and YMAX = 8E-03.
10. (A continuation of the discussion at the beginning of Section 3.) To see an example of a function whose graph will be assigned a tangent line at a point where the function is not differentiable, enter F(X) = ABS(X)/1E10 and accept the program's default parameter settings (with X0 = 0). The graph will appear as a straight line lying along the X-axis. When the display is complete, press |T| to see what happens.

D. Limit Problems

1. PURPOSE

This program provides practice in determining when limits exist and in finding them when they do.

2. DESCRIPTION

The program generates limit problems from a menu that includes polynomials, rational functions, roots, absolute values, infinite limits, and indeterminate forms. The program displays a limit problem and asks for the answer in the form of a number (e.g. 3, 1.5, or -4/5), an expression (e.g. SQR(5/7)), I or -I (for +/- infinity), or U (for undefined). The computer responds "RIGHT" for a right answer, and "WRONG" plus a hint for the first wrong answer. After two wrong answers, the computer displays the correct answer and the cumulative score of right answers as a percent of problems tried, and asks if you want another problem.

3. STEP BY STEP

Load the program from the main disk menu, read the greeting message, and press |RETURN| to display the menu of problem types (Screen 1).

<MENU OF PROBLEM TYPES>

1. POLYNOMIALS
2. RATIONAL FUNCTIONS
3. ROOTS
4. ABSOLUTE VALUES
5. INFINITE LIMITS
6. INDETERMINATE FORMS
7. A MIXTURE OF TYPES
8. QUIT

PRESS THE NUMBER OF THE PROBLEM TYPE YOU WANT.

Screen 1. Menu of problem types.

Typical problems are:

Polynomial: $\displaystyle \text{Limit}_{y \to -3} \; -y^2 + y - 5$

Rational: $\displaystyle \text{Limit}_{T \to 0} \; \frac{5*T^2 + 2*T}{3*T^3 - 2*T^2 - T}$

Root: $\displaystyle \text{Limit}_{y \to 25} \; \frac{5 - \sqrt{y}}{25 - y}$

Absolute value: $\displaystyle \text{Limit}_{y \to 5} \; \frac{|y - 5|}{3*y - 15}$

Infinite limit: $\displaystyle \text{Limit}_{x \to \infty} \; \sqrt{(25*x + 2)/(64*x - 4)}$

Indeterminate forms: $\displaystyle \text{Limit}_{y \to 0} \; \frac{\cos(-2*y)-1}{2*y}$

$\displaystyle \text{Limit}_{T \to \infty} \; \cos(-4/T)\wedge T$

If you press any of the keys |1| - |6|, the program will generate problems of the kind you choose. Mixtures of these programs are also available, and you may choose your own mixture by pressing |7|.

The problems listed above are samples of each type. There are several patterns within each type, and literally billions of different problems.

Press |7| to display the mixture menu.

```
                 <MENU OF PROBLEM TYPES>

              ? 1.   POLYNOMIALS
                2.   RATIONAL FUNCTIONS
                3.   ROOTS
                4.   ABSOLUTE VALUES
                5.   INFINITE LIMITS
                6.   INDETERMINATE FORMS

               |Y| to include a type in the mix
               |N| to exclude a type from the mix.
```

Screen 2. The mixture menu.

After reading the mixture menu, press |Y| or |N| at each option to make your selection.

The screens that follow are selected from a run we took through the program after we pressed |Y| six times to allow all types to be present in the problem mixture. Since the program generates problems randomly, your own excursion through the program is not likely to yield any of the examples shown here. These examples will, however, give you an idea of what to expect.

The first problem the computer displayed is shown in Screen 3. We deliberately gave a wrong answer. The computer responded with a hint (Screen 4). We then gave the right answer. The computer acknowledged, and displayed our score as .5 right out of 1 problem tried, or 50%.

```
                                                   |Screen 1
    1)  Limit  3*y + 10
        y->-3  3*y + 6
           = _

    Your answer should have the form—
      number . . . . 3 or 1.5 or -4/3
      expression . . SQR(5/7) or EXP(1)
      I or -I  . . . +/- INFINITY
      U  . . . . . . UNDEFINED
```

```
                                                   |Screen 2
    1)  Limit  3*y + 10
        y->-3  3*y + 6
           = 10/6              |WRONG|
           = _
    |HINT|   THE LIMIT OF THE DENOMINATOR
             IS NOT ZERO SO JUST EVALUATE
             THE FUNCTION.
```

```
                                                   |Screen 3
    1)  Limit  3*y + 10
        y->-3  3*y + 6
           = 10/6              |WRONG|
           = -1/3              |RIGHT|
    |HINT|   THE LIMIT OF THE DENOMINATOR
             IS NOT ZERO SO JUST EVALUATE
             THE FUNCTION.

             .5 RIGHT OF 1 TRIED = 50%
    Press |A|NOTHER  |M|ENU  OR  |Q|UIT  _
```

Screens 1-3 above show the progress through a problem answered incorrectly at first, and then correctly.

As the displays in Screens 1-3 suggest, the program always gives you two chances to answer. After that you may choose another problem (press |A|), change problem types

(press |M| for the problem type menu), or leave the program (press |Q| for "quit").

The next screen sequence shows how the program handles an improper input (Screen 4) and gives hints (Screens 5-8).

```
                                                  |Screen 4
      2)   Limit   3*y^2 + 12*y + 9
           y->-3   9*y^2 + 57*y + 90
                   = 0/0

      * THE PROGRAM CANNOT INTERPRET YOUR *
      * ANSWER.  PLEASE ANSWER AGAIN.     *
```

```
                                                  |Screen 5
      2)   Limit   3*y^2 + 12*y + 9
           y->-3   9*y^2 + 57*y + 90
                   = 1/3             |WRONG|
                   = _

      |HINT|    THE NUMERATOR AND DENOMINATOR BOTH
                APPROACH ZERO.  TRY TO FACTOR
                AND SIMPLIFY.
```

```
                                                  |Screen 6
      8)   Limit   -3*T^2 + T - 1
           T->-∞   3*T^2 + 3*T + 9
                   = I               |WRONG|
                   = _

      |HINT|    FACTOR T^2 FROM THE
                NUMERATOR AND DENOMINATOR.
```

Screen 7

10) Limit (1 - 3/y)^y
y->∞

= 0 WRONG
= _

HINT '1 ∞' INDETERMINATE FORM
REWRITE F^G AS EXP(G*LOG(F))
AND FIND THE LIMIT OF G*LOG(F).

Screen 8

11) Limit $\frac{\text{COS(2*x) - 1}}{\text{6*x}}$
x->0

= 2/6 WRONG
= _

HINT THE NUMERATOR AND DENOMINATOR
BOTH APPROACH ZERO. APPLY
L'HOPITAL'S RULE. (0/0 CASE)

Screens 4-8 show typical hints and error messages.

E. Limit Definition

1. PURPOSE

The ε-δ definition of limit is concise and powerful. But it is highly symbolic, and may be difficult to understand at first. The program LIMIT DEFINITION enables you to work with the definition graphically to help you interpret the symbols and understand what the definition really says.

2. THE DEFINITION OF LIMIT

Here is the formal definition of what it means for a real valued function F of a real variable X to have a limit L as X approaches a number C.

Definition of Limit

The limit of F(X) as X approaches C is the number L if:

For every $\varepsilon > 0$ (radius about L) there exists a $\delta > 0$ (radius about C) such that for all X

$$0 < |X - C| < \delta \quad \text{implies} \quad |F(X) - L| < \varepsilon.$$

In symbols, we write

$$\lim_{X \to C} F(X) = L$$

to say "the limit of F(X) as X approaches C equals L."

3. DESCRIPTION

The general pattern of the program is "given a function F(X), a point C, a suspected limit L as X->C, and an $\varepsilon > 0$, find a δ that satisfies the limit definition." The definition requires that for each positive epsilon there be a corresponding delta — your job in using this program is to find one graphically. If one value of δ satisfies the definition, then any smaller positive value will satisfy it as well. Thus, your job is really to find any one of a whole interval of suitable values. You should also look for the largest delta that satisfies the definition.

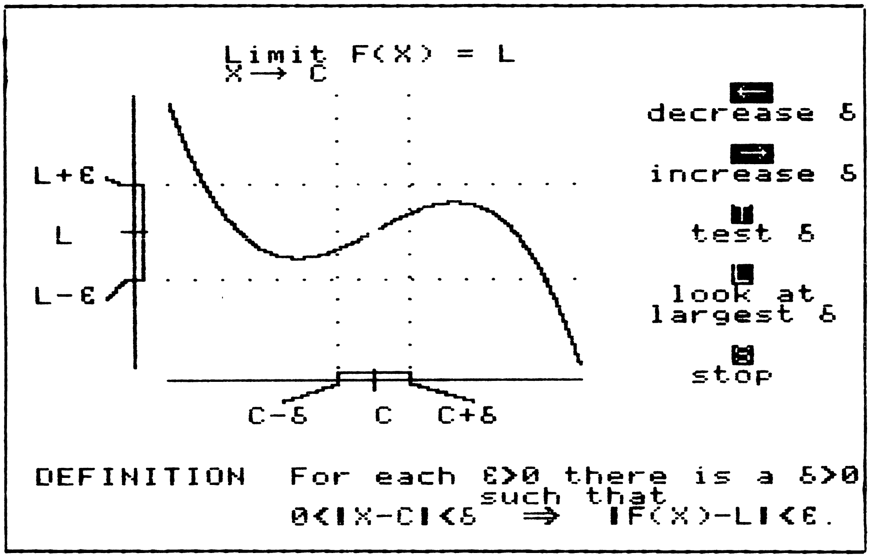
LIMIT DEFINITION is entirely graphical, and requires no calculation on your part. In preliminary tests of the program, most users reported that it did help them "see" what the definition said. The program will not help you calculate the value of L in a particular limit problem, however, nor will it help you find a numerical value of δ for a given ε. But it should give you insight that will be helpful as you solve numerical problems elsewhere.

4. STEP BY STEP

Select the program LIMIT DEFINITION from the main disk menu (it takes about fifty seconds to load) and press |RETURN| when you have read the greeting message and are ready to begin.

The computer will graph a function, label a point C on the X-axis and a point L on the Y-axis, and select a value of $\varepsilon > 0$. It will also draw horizontal dotted lines through

the points L $\pm$ ε on the Y-axis, choose a δ > 0 at random, and draw a pair of dotted vertical lines through the points C $\pm$ δ on the X-axis. The display should look something like the one shown in Screen 1.

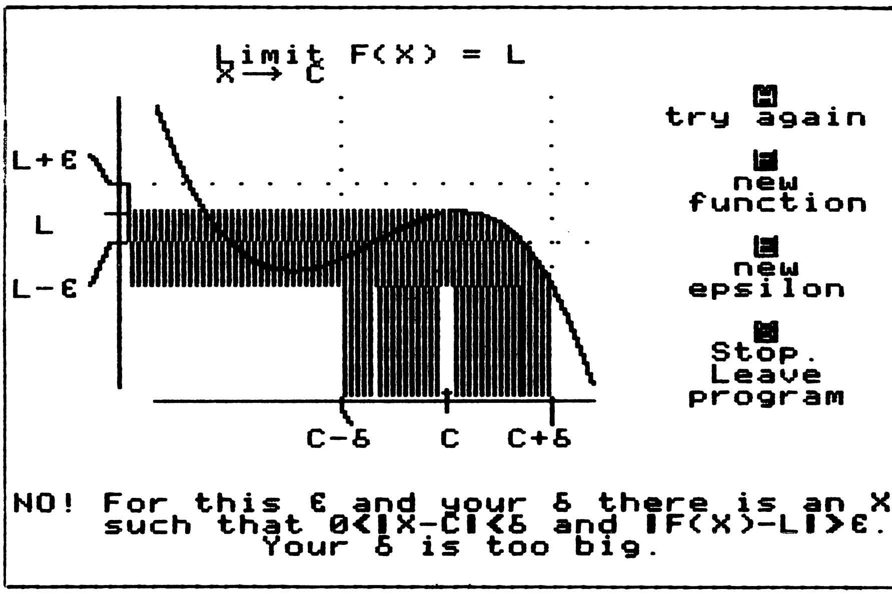


Screen 1. Find a δ for the given ε. Note the gap in the graph above the point C, a value excluded from consideration in the limit definition.

It is up to you to find a value of δ that satisfies the limit definition for the given C, F, L, and ε. The δ shown on the screen, which was chosen at random, may satisfy it already. If not, you can decrease δ by pressing |<-|. You can also increase δ (which you might do to find the largest suitable δ). You can proceed more quickly by holding down either arrow key (along with the |REPT| key, if necessary) or take larger steps by pressing |J| or |K|.

When you think you have a δ that satisfies the definition, press |T| to test it. The values of X between C − δ and C + δ (except for C itself) will be mapped by rays up to the curve and over to the Y-axis, and you can see if

all of the function values F(X) lie between L − ε and L + ε. A message from the computer will confirm what you see by telling you whether your δ is satisfactory and why. For instance, if your δ fails the test, your next display will look something like the one in Screen 2.



Screen 2. Testing a δ that is too large.

You now have four choices:

|A| Try again with the same value of ε. The graph will reappear as it was before you pressed |T|.

|E| Change the value of ε.

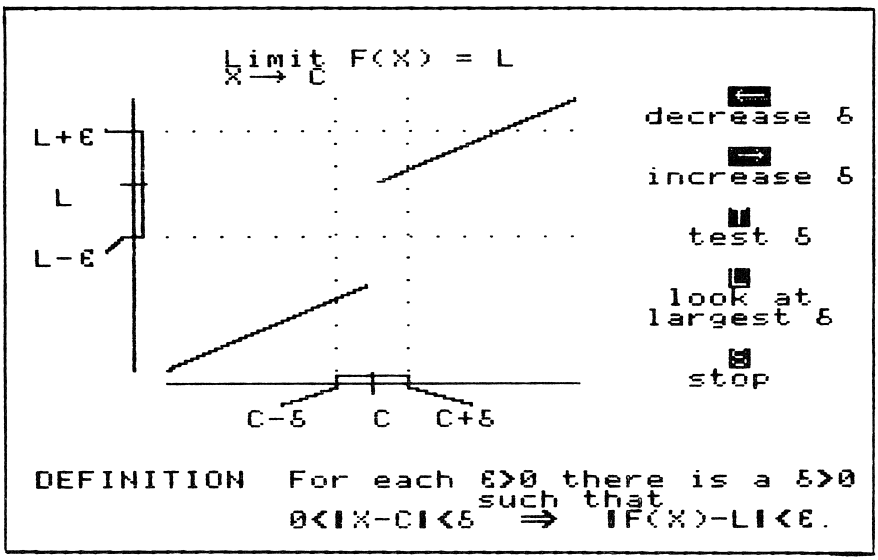
|F| Try a new function.

|Q| Quit the program and return to the main disk menu.

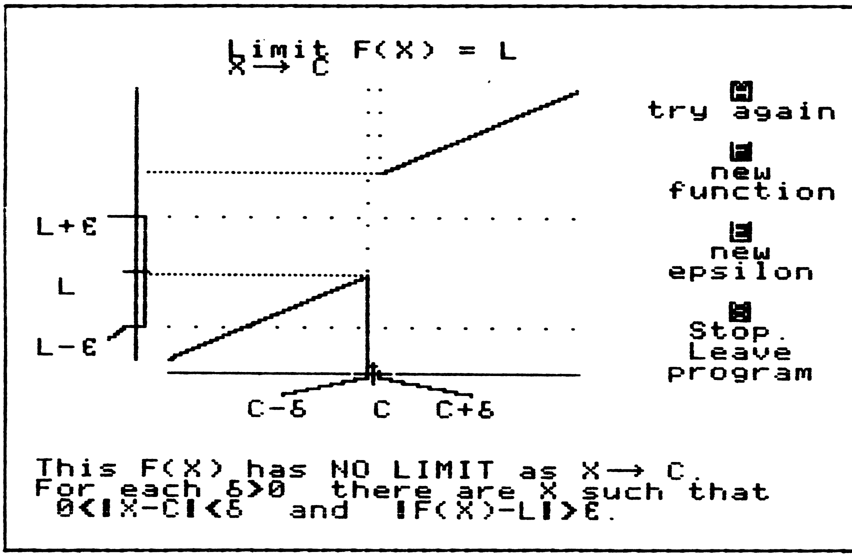
If you cannot find a value for δ that satisfies the definition, or if you cannot find the largest such δ, just press |L|. If the function has a limit as X->C, then the largest δ that satisfies the definition for the given ε will be shown. However, several functions in the program's inventory have no limit as X approaches certain values of C, and some ε's will have no corresponding δ in these cases.

5. ENCOUNTERING A POINT OF DISCONTINUITY

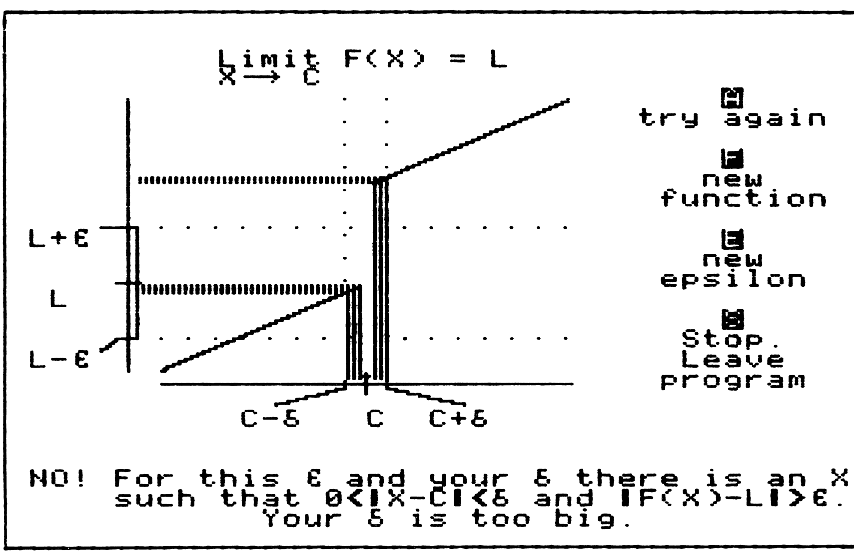
Screens 3, 4, and 5 show what can happen when you encounter a function that has no limit as X→C.



Screen 3. This function has no limit as X→C.

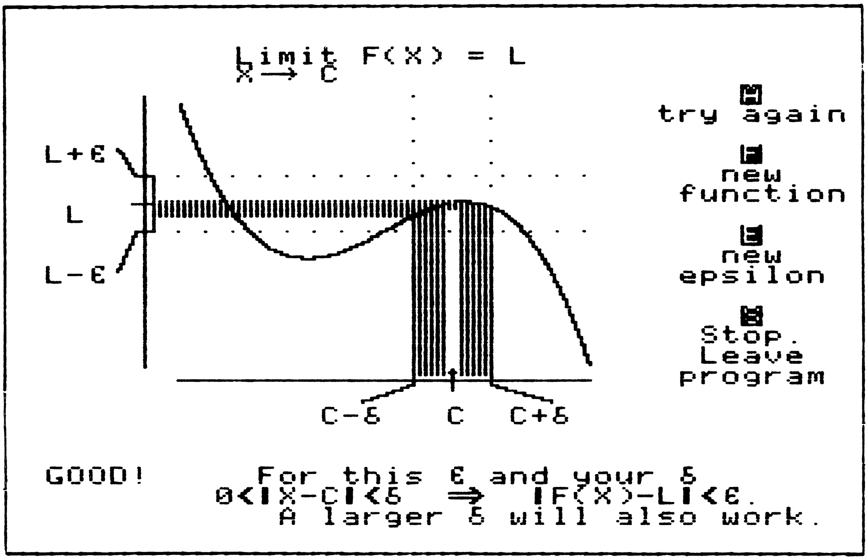


Screen 4. No positive delta is small enough.

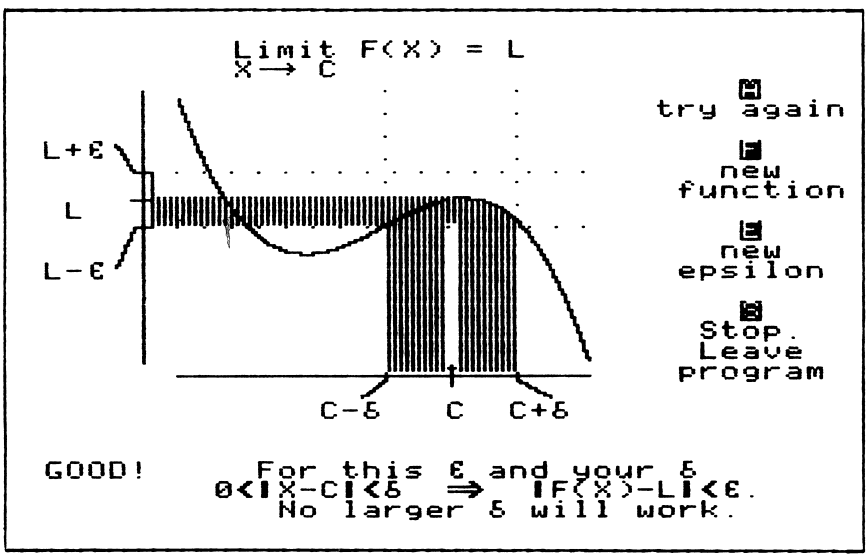


Screen 5. The response to a request for a largest delta when no limit exists.

6. SUCCESS AT LAST



Screen 6. A suitable delta for the problem, though not the largest delta.



Screen 7. The largest delta for this problem.

7. A PLEASANT SURPRISE

The program contains fifty combinations of F and C. You may ask for them by number (1 - 50) from any screen that has the |F| option (for example, Screen 2). Instead of requesting one of the options offered by the screen, press |*|. The prompt

TYPE NUMBER OF FUNCTION: _

will appear at the bottom of the screen. To try this out, go to Screen 2, press |*|, and then press |4| |5| |RETURN| to see what happens.

F. Continuity at a Point

1. PURPOSE

This program provides practice with the three-condition definition of continuity of a function at a point, with multi-line function definitions, and with one-sided limits.

2. DESCRIPTION

A function is defined on the screen by a two- or three-line definition and you are asked if the function is continuous at a particular point (Y or N). If your answer is correct, you may then see the graph, ask for another problem, or quit. If your answer is incorrect, you are led through the three conditions of the definition of continuity:

Is the function defined?
Does the limit exist?
Does the limit equal the function value?

The program then describes the continuity or discontinuity of the function in terms of the (correct) responses to these questions. The function's graph is available only after the three questions have been answered. After looking at the graph you may review the conditions, request another problem, or quit. The computer keeps score as you go along.

3. THE TEST FOR CONTINUITY AT A POINT

A real valued function F of a real variable X is continuous at a point C if it passes the following test:

The Continuity Test

The function Y = F(X) is continuous at X = C if and only if all three of the following statements are true:

1. F(C) exists (C lies in the domain of F).
2. $\lim_{X\to C} F(X)$ exists (F has a limit as X->0).
3. $\lim_{X\to C} F(X) = F(C)$.

In this test, the limit is a two-sided limit if C is an interior point of the domain of F; it is the appropriate one-sided limit if C is an endpoint of the domain.

To have a two-sided limit at an interior point C of its domain, a function F must have equal left-hand and right-hand limits at C. That is, the left- and right-hand limits

$$\lim_{X\to C-} F(X) \quad \text{and} \quad \lim_{X\to C+} F(X)$$

must exist at X = C and be equal. For F to be continuous at X = C, not only must these limits exist and be equal, but also F(X) must be defined at X = C and F(C) must equal the common value of these limits.

Example 1. For the function Y = F(X) graphed in Fig. 1 we have the following results:

a) F is continuous at X = 0 because
   1. F(0) exists (it equals 1),
   2. $\lim_{X\to 0+} F(X) = 1$ (F has a limit as X->0+),
   3. $\lim_{X\to 0+} F(X) = F(0)$ (the limit equals the function value).

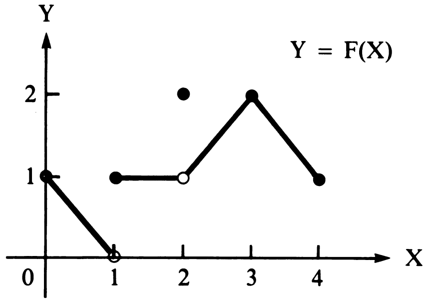


Figure 1. The function $Y = F(X)$ graphed here is discontinuous at $X = 1$ and $X = 2$.

b) F is not continuous at $X = 1$ because $\lim_{X\to 1} F(X)$ fails to exist. The function fails condition (2) of the test. (The right-hand and left-hand limits exist at $X = 1$, but they are not equal.)

c) F is not continuous at $X = 2$ because $\lim_{X\to 2} F(X) \neq F(2)$. The function fails condition (3) of the test.

d) F is continuous at $X = 3$ because
   1. $F(3)$ exists (it equals 2),
   2. $\lim_{X\to 3} F(X) = 2$ (F has a limit as $X\to 3$),
   3. $\lim_{X\to 3} F(X) = F(3)$ (the limit equals the function value).

e) F is continuous at $X = 4$ because
   1. $F(4)$ exists (it equals 1)
   2. $\lim_{X\to 4^-} F(X) = 1$ (F has a limit as $X\to 4$),
   3. $\lim_{X\to 4^-} F(X) = F(4)$ (the limit equals the function value).

To test for continuity, we always ask three questions:

1. Does $F(C)$ exist?
2. Does $\lim_{X\to C} F(X)$ exist?
3. Does $\lim_{X\to C} F(X) = F(C)$?

For F to be continuous at $X = C$, all three answers must be yes.

4. STEP BY STEP

Load the program, read the greeting message, and press |RETURN| to request a problem. The problem will have a problem number (in this case (1) because it is your first) and the program will wait for you to answer the question

IS F(X) CONTINUOUS AT X = <some number>? (Y/N)

with a yes or no by pressing |Y| or |N|.

The program selects problems randomly, so your problem sequence will probably not be exactly like the one that follows here. However, press |Y| or |N| when your first problem comes up, and go on to see what happens.

Here is a record of what happened when we started through the program while writing this chapter.

The first problem to come up was:

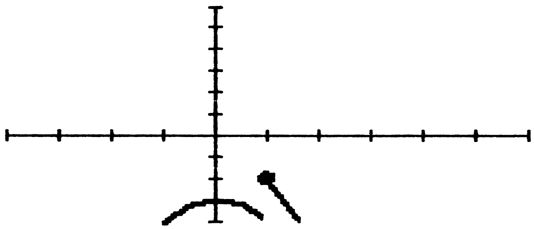
```
1)   F(X)= { -1 - 2*X   IF X ≠ -2
           { 2          IF X = -2
                                       |WRONG|
IS F(X) CONTINUOUS AT X = -2? (Y/N)  |Y|
---------------------------------------------
1. IS F(-2) DEFINED? (Y/N) |Y| |RIGHT|
      F(-2) = 2
                                       |RIGHT|
2. DOES LIM(X->2) F(X) EXIST? (Y/N)  |Y|

LIM(X->-2-) F(X)= 3   SO LIM(X->-2) F(X)
LIM(X->-2+) F(X)= 3            = 3
3. DOES LIM(X->-2) F(X) = F(-2)? (Y/N) |Y|
                      3 ≠ 2          |WRONG|
---------------------------------------------
F(X) DOES NOT SATISFY CONDITION 3, SO
F(X) IS NOT CONTINUOUS AT X = -2

Press |A|nother  or |G|raph  or |Q|uit
```

After learning what we did wrong, we pressed |A| to request another problem:

```
SCORE:  0  RIGHT OF 1 TRIED = 0 %
2)   F(X)= ⎧-3 - X*X  IF X < 1
           ⎨-2        IF X = 1
           ⎩1         IF X > 1
                                         |WRONG|
IS F(X) CONTINUOUS AT X = 1? (Y/N)   |Y|
------------------------------------------------
1. IS F(1) DEFINED? (Y/N)   |Y|  |RIGHT|
      F(1) = -2
                                         |WRONG|
2. DOES LIM(X->1) F(X) EXIST? (Y/N)  |Y|

LIM(X->1-) F(X)= -4   SO LIM(X->1) F(X)
LIM(X->1+) F(X)= -2          =  UNDEFINED
------------------------------------------------
F(X) DOES NOT SATISFY CONDITION 2, SO
F(X) IS NOT CONTINUOUS AT X = 1

Press |A|nother  or |G|raph  or |Q|uit
```

After this mistake, we pressed |G| to view the graph:

```
SCORE:  0 RIGHT OF 1 TRIED = 0 %
2)    F(X)= ┌-3 - X*X IF X < 1
            │-2       IF X = 1
            └1 - 3*X  IF X > 1
                                      |WRONG|
IS F(X) CONTINUOUS AT X = 1? (Y/N)    |Y|
```



```
F(X) DOES NOT SATISFY CONDITION 2, SO
F(X) IS NOT CONTINUOUS AT X = 1

Press |A|nother or |C|onditions or |Q|uit
```

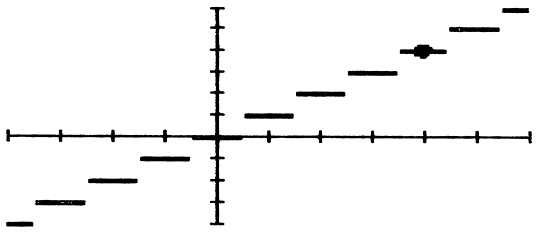
We answered the next problem correctly, and asked for the graph:

SCORE: 0 RIGHT OF 2 TRIED = 0 %

3) $F(X)= \begin{cases} INT(X + .5) & \text{IF } X \neq 4 \\ 4 & \text{IF } X = 4 \end{cases}$

IS F(X) CONTINUOUS AT X = 1? (Y/N) |Y|

|RIGHT|



Press |A|nother or |C|onditions or |Q|uit

We answered the next one incorrectly . . .

```
SCORE:  1 RIGHT OF 3 TRIED = 33.3%
4)    F(X)= {1 + ABS(X + 2)   IF X ≠ -1
            {3                 IF X = -1
                                     |WRONG|
IS F(X) CONTINUOUS AT X = -1? (Y/N) |Y|
---------------------------------------------
1. IS F(-1) DEFINED? (Y/N)  |Y|  |RIGHT|
      F(-1) = 3
                                         |RIGHT|
2. DOES LIM(X->-1) F(X) EXIST? (Y/N)  |Y|

LIM(X->-1-) F(X)= 2   SO LIM(X->-1) F(X)
LIM(X->-1+) F(X)= 2            = 2
3. DOES LIM(X->-1) F(X) = F(-1)? (Y/N) |Y|
                          2 ≠  3        |WRONG|
---------------------------------------------
F(X) DOES NOT SATISFY CONDITION 3, SO
F(X) IS NOT CONTINUOUS AT X = -1

Press |A|nother  or |G|raph  or |Q|uit
```

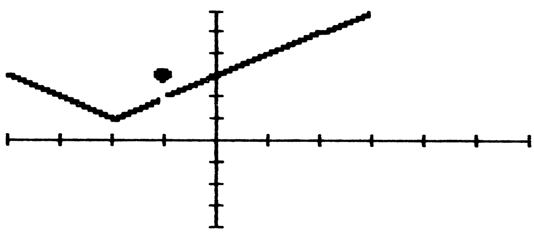
and asked for the graph:

SCORE: 0 RIGHT OF 3 TRIED = 33.3%

4) F(X)= $\begin{cases} 1 + \text{ABS}(X + 2) & \text{IF } X \neq -1 \\ 3 & \text{IF } X = -1 \end{cases}$

|WRONG|

IS F(X) CONTINUOUS AT X = -1? (Y/N) |Y|



F(X) DOES NOT SATISFY CONDITION 3, SO
F(X) IS NOT CONTINUOUS AT X = -1

Press |A|nother or |C|onditions or |Q|uit

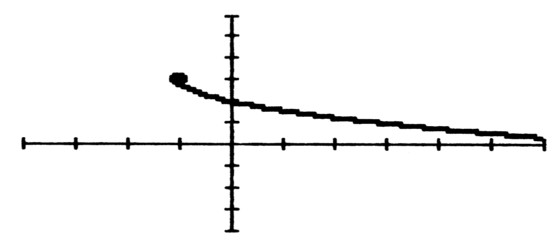
The next problem was about the behavior of a function at the left-hand endpoint of its domain. We answered it correctly, and asked to see the graph:

SCORE: 2 RIGHT OF 6 TRIED = 33.3%

7) $F(X)= \begin{cases} 3 + SQR(X + 1) & \text{IF } X \neq -1 \\ 3 & \text{IF } X = -1 \end{cases}$

IS F(X) CONTINUOUS AT X = -1? (Y/N) |Y|

|RIGHT|



Press |A|nother or |C|onditions or |Q|uit

The displays we have chosen to duplicate here do not show it, but you will notice when you work on your own that the program will require you to go through the three conditions of continuity even when your answer to the first question is correct.

After you have pressed |G| to graph a function, you may press |C| to return to the display of continuity conditions. Try it at some point. After the condition statements return to the screen you may recall the graph by pressing |G|. It will appear quickly this time because it does not have to be redrawn. Thus you may "toggle" back and forth between conditions and graph to see how they compare.

G. Derivative Grapher

1. PURPOSE

This program enables you to study relationships among the functions

$$F,\quad F',\quad F'',\quad \text{and} \quad \int_A^X F(t)dt$$

by graphing any two of them simultaneously over an interval $A \leq X \leq XMAX$ of your choice.

2. ON THE SCREEN, $\int F(t)dt$ MEANS $\int_A^X F(t)dt$

The usual notation for the definite integral of F(t) from t = A to t = X will not fit on a single screen display line, and space constraints forced us to omit the limits of integration on some screens. All integrals in this program are, nevertheless, definite integrals.

3. STEP BY STEP

It takes about 45 seconds to load DERIVATIVE GRAPHER from the main disk menu. After reading the greeting message, press |RETURN| to display the function menu shown in Screen 1. This is the menu for choosing functions to study. You may enter a function of your own (select option #12), or use one of the given functions.

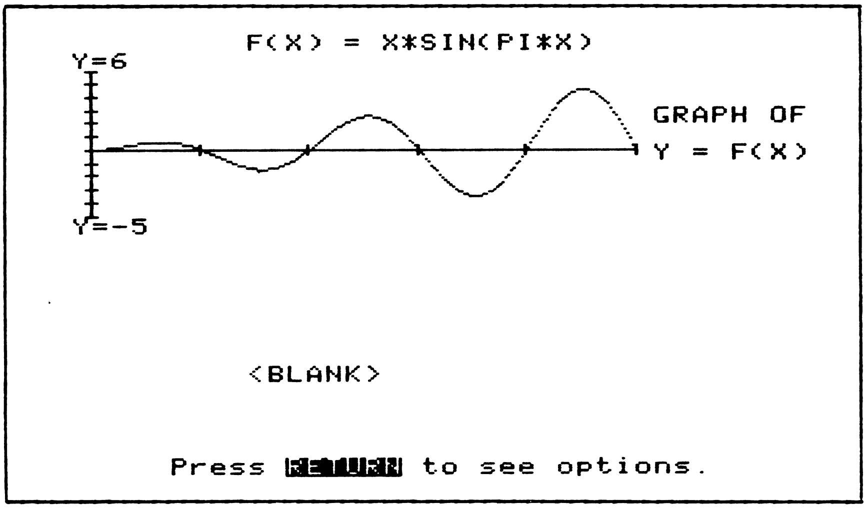
Select F(X) = X*SIN(PI*X) from the function menu by pressing |1| |1| |RETURN|. The function will be graphed in the window $0 \leq X \leq 5$, $-5 \leq Y \leq 6$, as shown in Screen 2.

The graphics display in this program is always divided into two parts. Each part can contain a graph of F(X), F'(X), F"(X), or $\int$F(X)dx so that you can see the graphs of any two of these functions simultaneously in either order. A function selected from the function menu will be graphed in the top part, leaving the bottom part blank for you to fill in (more about this in a moment).

After examining the graph of F(X) = X*SIN(PI*X), press |RETURN| to display the main options menu (Screen 3).

```
|<FUNCTIONS AVAILABLE>|

1   X*X
2   X*X-1
3   X*X-4
4   -X*(X-4)
5   X^3
6   X*(X-1)*(X-3)
7   X*X*(X+1)*(X-2)
8   ABS(X)+1
9   SIN(X)
10   SIN(X)+1
11   X*SIN(PI*X)
12   DEFINE YOUR OWN

Type a number then press |RETURN|.
```

Screen 1. The function menu.



Screen 2. The graph of F(X) = X * SIN(PI*X).

|<CURRENT DISPLAY>|

F(X) = X*SIN(PI*X)
TOP Y = F(X)
BOTTOM <BLANK>
X FROM 0 TO 5

|<OPTIONS>|

1 RETURN TO DISPLAY
2 CHANGE TOP DISPLAY
3 CHANGE BOTTOM DISPLAY
4 CHANGE FUNCTION
5 CHANGE GRAPHING SCALES
6 QUIT

Press the number of your choice.

Screen 3. The main options menu.

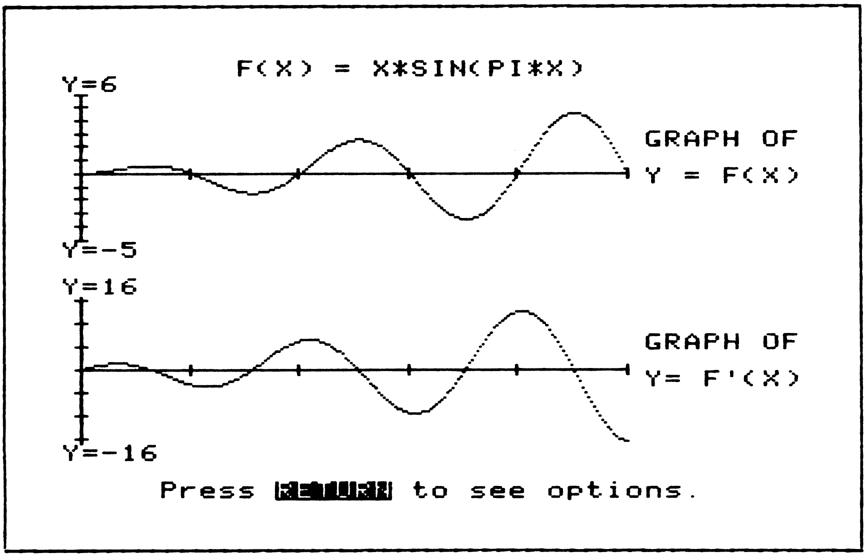
To proceed from the main options menu, press

|1| to recall the most recent graphics display
|2| for a menu for changing the top display
|3| for a menu for changing the bottom display
|4| to return to the function menu
|5| to rescale the graphs of F, F', F", or $\int F(t)dt$
|6| to quit the program.

For example, to graph F'(X) at the bottom of the graphics display, press |3| on the main options menu, and then |2| when the secondary options menu (Screen 4) appears. The graphics display will now show the graph of F' below the graph of F, as in Screen 5.

```
                  |<CURRENT DISPLAY>|

        F(X) = X*SIN(PI*X)
        TOP        Y = F(X)
        BOTTOM     <BLANK>
          X FROM   0 TO 5

     REPLACE BOTTOM DISPLAY WITH
             1    GRAPH OF  Y = F(X)
             2    GRAPH OF  Y = F'(X)
             3    GRAPH OF  Y = F"(X)
             4    GRAPH OF  Y = ∫F(t)dt
             5    <BLANK>
             6    KEEP CURRENT DISPLAY

        Press the number of your choice.
```

Screen 4. The secondary options menu for the bottom display.



Screen 5. The result of pressing |2| in Screen 4.

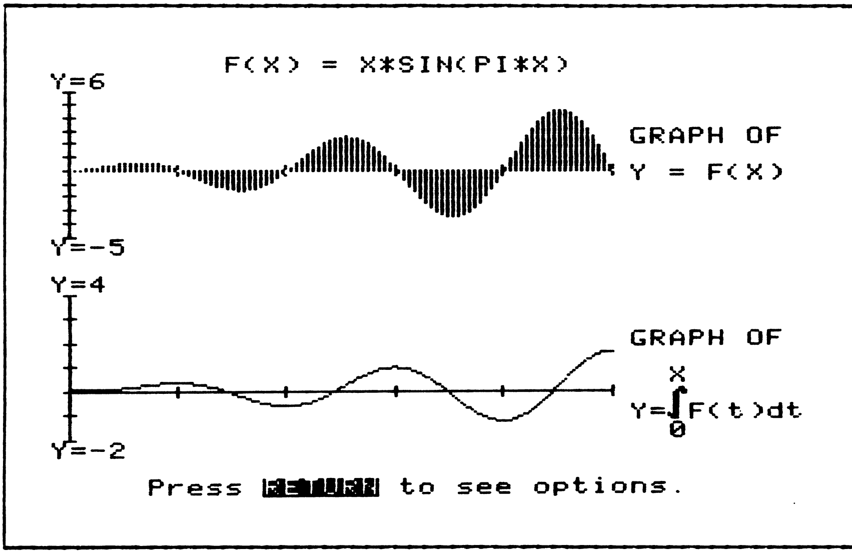
To replace the bottom display in Screen 5 by the graph of

$$\int_0^X F(t)dt,$$

press |RETURN| (for the main options menu), |3| (to change the bottom display), and |4| (to graph $\int F(t)dt$). The graph of the integral of F from 0 to X will appear on the bottom half of the screen as the region between the graph of F and the X-axis on the top half of the screen fills with color. Screen 6 on the next page shows the completed display.

4. GRAPHING SCALES

The functions F, F', F", and $\int F(t)dt$ in any example are always graphed over a common interval XMIN $\leq$ X $\leq$ XMAX. This interval is set in advance for functions 1-11 on the function menu, but it may be changed by pressing |5| on



Screen 6. The graph of F is "filled" as its integral is graphed.

F(X) = X*SIN(PI*X)

|<CURRENT SCALES FOR GRAPHS>|

1	ALL	X = 0	TO 5
2	F(X)	Y = -5	TO 6
3	F'(X)	Y = -16	TO 16
4	F"(X)	Y = -50	TO 50
5	∫F(t)dt	Y = -2	TO 4

Press |RETURN| to keep these values or press the number to be changed. _

Screen 7. The change scale menu, showing the program's default values for F(X) = X*SIN(PI*X).

the main options menu and entering new endpoint values when the graphing scale menu appears. For F(X) = X*SIN(PI*X), the menu looks like the one in Screen 7.

The change scale menu enables you to use different vertical scales for the graphs of F, F', F", and $\int F(t)dt$.

The menu also enables you to change the horizontal scale by pressing |1| and entering new values for XMIN and XMAX followed by |RETURN|s. Whenever you do this, F(X) will automatically be regraphed in the top display and the bottom display will once again be blank.

When you define your own function F(X) by pressing |1| |2| on the function menu and entering a formula for F(X) followed by a |RETURN|, you are immediately given an opportunity to enter values for XMIN, XMAX, YMIN, and YMAX for the graph of F(X). The values you enter for YMIN and YMAX will define the vertical scales for the graphs of F', F", and $\int F(t)dt$ as well, unless you choose to alter these values with the change scale option.

PROBLEMS

The following problem sections involve a "guided discovery" of some of the relations among F, F', F" and $\int F(t)dt$. You will be asked to fill in tables with information about particular graphs. You will then be asked to draw tentative conclusions from this information and to check these conclusions against other graphs.

Part A: F and F'

For each function in Problems 1-3, use DERIVATIVE GRAPHER to display F and F' together, then complete the table. It is difficult to read the values of F(X) and F'(X) from the graphs precisely, but in most cases all you need to record is whether the value is positive (+), negative (-), or zero (0).

1. F(X) = X*X - 4 (menu function #3). Copy and complete the table. At what value of X is F(X) a maximum? Add this X value to your table.

X	F increasing decreasing	F(X)	F'(X)
-2			
-1	D	-	-
1	I	-	+
3			

2. Copy and complete the table for

   F(X) = X*(X - 1)*(X - 3)

   (menu function #6). Also add to the table the X values at which relative maxima and minima of F occur.

X	F increasing decreasing	F(X)	F'(X)
0	I	0	+
1			
2			
3			
4			
1.5			

3. Copy and complete the table for F(X) = SIN(X) (menu function #9).

X	F increasing decreasing	F(X)	F'(X)
1			
1.6			
3.1			
4			
6.3			
7.9			

Use the tables completed in Problems 1-3 to help answer the questions in Problems 4-13.

4. If F(X) is increasing at X then F'(X) is ___. (+, -, 0)
5. If F(X) is decreasing at X then F'(X) is ___. (+, -, 0)

6. If F'(X) is positive at X, then, near X, F is ___.
   (a) increasing, (b) decreasing, (c) could be either.
7. If F(X) has a maximum at X then F'(X) is ___. (+, -, 0)
8. If F(X) has a minimum at X then F'(X) is ___. (+, -, 0)
9. If F'(X) = 0 at X, F
   (a) must have a maximum value, (b) must have a minimum value, (c) must have either a maximum or a minimum value, (d) may still fail to have either a maximum or a minimum value.
10. If F(X) > 0, then F'(X) ___. (a) must be +, (b) must be -, (c) must be 0, or (d) could be any number
11. If F(X) < 0 then F'(X) ___. (a) must be +, (b) must be -, (c) must be 0, or (d) could be any number
12. If F(X) = 0 then F'(X) ___. (a) must be +, (b) must be -, (c) must be 0, or (d) could be any number
13. If F'(X) > 0 then F(X) ___. (a) must be +, (b) must be -, (c) must be 0, or (d) could be any number
14. Make similar tables for F(X) = X*X*X and F(X) = -X * (X - 4) to see if they support your answers to the previous questions.

Part B: F"(X)

To each of the tables in Part A, add a column labeled F"(X). Fill in this column with +, -, or 0. Then use this information to help answer the questions in Problems 15-20.

15. If F(X) > 0, can F"(X) be
    (a) positive? (b) negative? (c) zero?
16. If F(X) < 0, can F"(X) be
    (a) positive? (b) negative? (c) zero?
17. If F'(X) > 0, can F"(X) be
    (a) positive? (b) negative? (c) zero?
18. If F'(X) < 0, can F"(X) be
    (a) positive? (b) negative? (c) zero?
19. If F has a relative maximum at X, can F"(X) be
    (a) positive? (b) negative? (c) zero?
20. If F has a relative minimum at X, can F"(X) be
    (a) positive? (b) negative? (c) zero?

Part C: $\int F(t)dt$

21. If a function F is continuous on an interval [A,B], and X is a point in [A,B], then the integral

$$I(X) = \int_A^X F(t)dt$$

is a differentiable function of X throughout [A,B]. We therefore know that F and its integral I(X) are related in the following ways:

i) If I has a relative maximum or minimum value at a point X = C between A and B, then F(C) = 0.
ii) I is an increasing function of X on any interval on which F is positive.
iii) I is a decreasing function of X on any interval on which F is negative.

a) Look at Screen 5 to see these three relationships in the graphs there.
b) Look for these relationships in the graphs of the other functions on the function menu.

H. Function Evaluator/Comparer

1. PURPOSE

This program computes the values of one or two functions at single points or at a preselected number of points (1 to 500) in an interval. This facilitates graphing, enables you to make informed guesses about the locations of zeros and extrema of functions, and indicates when two functions may be identical or differ by a constant. If you are using two functions, the program also computes the maximum and minimum values of the difference of the functions at the selected input points.

2. DESCRIPTION

Once the program is loaded and you have cleared the greeting message from the screen, you may choose to work with one function, F(X), or two, F(X) and G(X). The default functions are F(X) = SIN(X) and G(X) = COS(X). You can keep them both by pressing |2| and two |RETURN|s, or you can press |1| or |2| and type in your own function or functions, with each entry followed by a |RETURN|. You will then be asked if you want to compute the values of the functions at individual

X-values, which you will supply (press |P| for "point mode"), or at equally spaced X-values on some interval (press |I| for "interval mode").

If you select the point mode, the screen will be cleared and labeled with F(X) and G(X), and you will be asked to enter values for X one at a time, followed by |RETURN|s. The program will display the values of X, F(X), and G(X). In point mode, the program evaluates your functions only at the X-values you specify. To stop, press |Q| |RETURN|. The program will then print the maximum and minimum values computed for F(X), G(X), and F(X) - G(X).

If you select the interval mode, you will be asked to enter the endpoints of the interval and to specify the number of equally spaced points in the interval. The program will then display the values X, F(X), and G(X) as X steps through the interval.

You can halt the computations by pressing the space bar and then resume them by pressing the space bar again. You can stop the computations completely by pressing |ESC|.

When F(X) and G(X) have been shown for all X values, press any key to display the maximum and minimum values computed for F(X), G(X), and F(X) - G(X).

Finally, you have a choice of repeating the calculations (press |R|), keeping the same functions but with a different mode or interval (press |K|), rerunning the program from the beginning (press |B|), or leaving the program (press |Q|).

3. STEP BY STEP

Load the program from the main disk menu, read the greeting message, and press |RETURN| to display the prompt

NUMBER OF FUNCTIONS? (1 OR 2) = |_|

Press |2|, and accept the default functions F(X) = SIN(X)

and G(X) = COS(X) by pressing |RETURN|s. A list of computing options will then appear, and the completed display will look like the one in Screen 1.

```
                 |<EVALUATOR/COMPARER>|
     NUMBER OF FUNCTIONS?  (1 OR 2) = 2

     F(X) = SIN(X)
     G(X) = COS(X)

                      |<OPTIONS>|
     COMPUTE F(X) AND G(X) :
                   P   AT SINGLE POINTS
                   I   ON AN INTERVAL
                   C   CHANGE FUNCTION(S)
                   Q   QUIT
                   PRESS P OR I OR C OR Q
```

Screen 1. The function display and option menu.

Press |I| for interval mode, and accept the default values

XMIN = 0
XMAX = 6.28318531

by pressing |RETURN|s. Then press |4| |3| |RETURN| to enter 43 for the number of steps. This will sample the functions at 44 equally spaced points, starting at the left-hand endpoint, X = 0. Pressing this last |RETURN| also begins the production of a table of values that scrolls up the screen and comes to rest displaying the last eight lines. When the computations stop, the display will look like the one in Screen 2.

```
X = 0 to 2*PI
F(X) = SIN(X)                    IN 43 STEPS
G(X) = COS(X)

5.26034119    -.85359309    .52094034
5.40646178    -.76864714    .63967302
5.55258237    -.66731881    .74477218
5.69870295    -.55176774    .83399782
5.84482354    -.4244567     .90544824
5.99094413    -.2880991     .9576006
6.13706472    -.14560116    .98934336
6.28318531    0             .99999999

PRESS ANY KEY FOR
MINIMA AND MAXIMA.
```

Screen 2. The last eight lines of the interval-mode table generated for SIN(X) and COS(X).

To continue the demonstration, press any standard key to display the maximum and minimum values computed in this table for the functions SIN(X), COS(X),and SIN(X) - COS(X). The display will then change to the one shown in Screen 3.

Warning: Do not confuse the computed maxima and minima with the maximum and minimum values assumed by the functions on the interval $0 \leq X \leq 2*PI$. They are different in this example (the true values are 1 and -1), as is often the case. They are the maximum and minimum of the values <u>computed at the sampled points</u>. The sampled values of X and the subsequent computations are vulnerable to truncation and round-off errors as well.

```
X = 0 to 2*PI
F(X) = SIN(X)                    IN 43 STEPS
G(X) = COS(X)
COMPUTED MINIMA:
F(X)= -.999332848        AT X=4.67585
G(X)= -.997332284        AT X=3.06853
F(X)-G(X)= -1.41209099 AT X=5.55258

COMPUTED MAXIMA:
F(X)= .999332848         AT X=1.60732
G(X)= 1                  AT X=0
F(X)-G(X)= -1.41397767 AT X=2.33792

|R|EDO COMPUTATIONS
|K|EEP F (AND G), NEW POINTS/INTERVAL
|B|EGIN PROGRAM AGAIN
|Q|UIT, LEAVE PROGRAM
```

Screen 3. The maximum and minimum values computed for F, G, and F - G on the interval 0 ≤ X ≤ 2*PI.

Press |B| to begin the program again, then press |1| and enter the single function

F(X) = (1 - COS(X))/X

Press |P| for point mode and enter the X values shown in Screen 4 one at a time to complete the display.

When you have entered the values of X shown in Screen 4 and noted the resulting values of F, press |Q| |RETURN| to request the maximum and minimum values from the computation. The display will change to the one in Screen 5.

```
                 POINT MODE
F(X) = (1-COS(X))/X
   X              F(X)

5                .14326756
1                .45969769
.2               .09966711
.04              .01999733
7E-04            3.5023E-04
0                UNDEFINED
-3               -.66333084
-.20000001       -.09966712
-5.0001E-04      -2.5028E-04
         X =
   TYPE A NUMBER AND PRESS RETURN.
  TO STOP, TYPE Q AND PRESS RETURN
```

Screen 4. Point mode. Note "UNDEFINED" at X = 0.

```
                 POINT MODE
F(X) = (1-COS(X))/X

COMPUTED MINIMUM:
F(X) = -.663330832    AT X=-3

COMPUTED MAXIMUM:
F(X) = .459697694     AT X=1
 |R|EDO COMPUTATIONS
 |K|EEP F (AND G), NEW POINTS/INTERVAL
 |B|EGIN PROGRAM AGAIN
 |Q|UIT, LEAVE PROGRAM
```

Screen 5. The maximum and minimum values from the list in Screen 4.

4. USES OF FUNCTION EVALUATOR

4.1 Creating a table for analysis and graphing

FUNCTION EVALUATOR enables you to examine the numerical behavior of a function easily and quickly. If the points are irregularly spaced, use the point mode. If the points are regularly spaced or if you need a table of function values, use the interval mode.

4.2 Estimating limits

FUNCTION EVALUATOR can sometimes help you to estimate or discover limiting values of functions. If you want to investigate the possibility that a function F(X) has a limit as X approaches C, enter F(X), select the point mode, and type in values of X that are close to C. The resulting table may give valuable information about the limiting behavior of the function.

4.3 Estimating maxima and minima

FUNCTION EVALUATOR lets you search for extrema in a crude but often effective way—by examining the values of the function at a large number of points. The program keeps track of the maximum and minimum values it has computed and their locations.

TWO WARNINGS: (i) There is no guarantee that an extremum will occur at any of the points at which F(X) was calculated. (ii) The program prints only the first (leftmost) occurrence of an extremum and there may be others.

4.4 Does F(X) = G(X) for all X?

What do you do when your solution to a textbook calculus problem is a function given by a formula that does not resemble the one given in the answer section? Sometimes it is clear how to change one of the formulas into the other with algebra or trigonometry, but not always. Even when your answer is correct, the formulas may not appear to represent the same function. For example, the formulas

$$f(x) = \sec x + \tan x \quad \text{and} \quad g(x) = \tan(\frac{\pi}{4} + \frac{x}{2})$$

represent the same function of x, but would you be willing to put effort into showing this (or even believe it) without some kind of preliminary evidence?

FUNCTION EVALUATOR provides two ways to compare two functions F(X) and G(X) at a large number of points.

If the formulas for F and G are relatively short, type in F(X) - G(X) as a single function. Choose an appropriate interval, select a hundred or more comparison points, and scroll through the values of F(X) - G(X). If the values are all zero or close to it (some might be 1E-08, for example), then F(X) may actually be equal to G(X) for all X in the interval and a more thorough investigation of the possibility of equality is warranted.

If the formulas for F and G are too long to allow entering F(X) - G(X), you may enter F and G separately, scroll through their values, pausing whenever you want, and then check the max-min screen for the largest and smallest values computed for F(X) - G(X). These data together should indicate whether the question of function equality is worth exploring further.

If you compare the values of F(X) = 2*SIN(X)*COS(X) and G(X) = SIN(2*X) in 100 steps through the interval 0 $\leq$ X $\leq$ 2*PI by entering F(X) - G(X) as a single function, the function values will all be 0 or -1E-08. The possibility that F(X) = G(X) for all X is definitely worth exploring.

If you enter these functions separately and sample over the same interval at the same points, the table will show identical values for F and G and the max-min screen will show the minimum and maximum values of F(X) - G(X) to be

-2.96859071E-09 at X = 4.58672

and

9.31322575E-10 at X = 2.13628

Again, it seems likely that F = G.

If you find a particular F(X) and G(X) to be equal for one subdivision of an interval, try other subdivisions as well. The functions F(X) = SIN(20*X) and G(X) = SIN(40*X) will produce equal values if you sample the interval 0 ≤ X ≤ 2*PI in 20 steps, but not if you sample at a larger number, or at a smaller number such as 3.

4.5 Does F(X) = G(X) + C for all X?

When you are evaluating indefinite integrals, it is useful to know when two functions F(X) and G(X) differ by a constant.

Compute F(X) - G(X) for a large number of X values in an interval. If the computed values of F(X) - G(X) are approximately constant, then it may be worth investigating to see whether F(X) = G(X) + C for all X in the interval.

PROBLEMS

Estimate the limits in Problems 1 - 5.

1. $\lim_{X \to 0}$ 2*X/(X + 7*SQR(X))
2. $\lim_{X \to 0}$ (1 - COS(X))/(X*X)
3. $\lim_{X \to 0}$ (SIN(X)/(EXP(X) - 1)
4. $\lim_{X \to \infty}$ (X - SQR(X*X + X))
5. $\lim_{X \to 0+}$ X*LOG(X)
6. Estimate the minimum value of F(X) = (2*X^3)/(2*X - 8.5), 4.25 < X < 8.5
7. Is the function 3 + 4*COS(X) + COS(2*X) ever negative?
8. The equation SQR(X) + SQR(X + 1) = 4 is known to have a solution in the interval [3,4]. Estimate it.
9. Estimate the largest absolute error in each of the following approximations on the interval -.1 ≤ X ≤ .1.
   a) 1/(1 - X) ≐ 1 + X
   b) 1/(1 + X) ≐ 1 + X + X*X
   c) SQR(1 + X) ≐ 1 + (X/2)
   d) SQR(1 + X) ≐ 1 + (X/2) - (X*X/8)

Test the functions in Problems 10–17 to indicate whether they differ by a constant on some domain of X-values. If you think they do, find the constant, and verify the result with algebra. What is the largest possible domain in each case?

10. $\dfrac{x}{x+1}$ and $\dfrac{-1}{x+1}$

11. $\dfrac{x^2+3}{x^2+1}$ and $\dfrac{2}{x^2+1}$

12. $\ln 2x$ and $\ln 3x$

13. $\tan x \sin 2x$ and $-2\cos^2 x$

14. $\sin x$ and $\sqrt{\sin^2 x}$

15. $\sin^2 x$ and $-\cos^2 x$

16. $\sin^2 x$ and $\cos^2 x$

17. $\sin^2 x$ and $-\frac{1}{2}\cos 2x$

18. Compare the values of

$f(x) = \sec x + \tan x$ and $g(x) = \tan(\frac{\pi}{4} + \frac{x}{2})$

at a large number of points.

I. Parametric Equations

1. DESCRIPTION

This program graphs parametric equations of the form

X = X(T), Y = Y(T)

by showing three graphs on the screen:

X vs. T, Y vs. T, and Y vs. X.

You enter X(T) and Y(T) and an interval for T, and the computer does the rest. The three graphs are traced simultaneously by moving particles as T runs through the parameter interval from TMIN to TMAX. Thus, you can see motion in the XY-plane resolved into components in the XT- and YT-planes. You may stop and restart the motion at any time.

The program also offers the option of viewing full-screen graphs of X(T), Y(T), and Y vs. X separately. To graph Y vs. X, the computer plots the point pairs (X(T),Y(T)).

2. STEP BY STEP

After loading the program, read the greeting messages and go on to the opening menu shown in Screen 1.

```
COORDINATE FUNCTIONS
  X(T) = COS(T)
  Y(T) = SIN(T)

T-DOMAIN
  TMIN = 0
  TMAX = 6.28318531

|C|HANGE ENTRY     |G|O ON     |Q|UIT  |_|
```

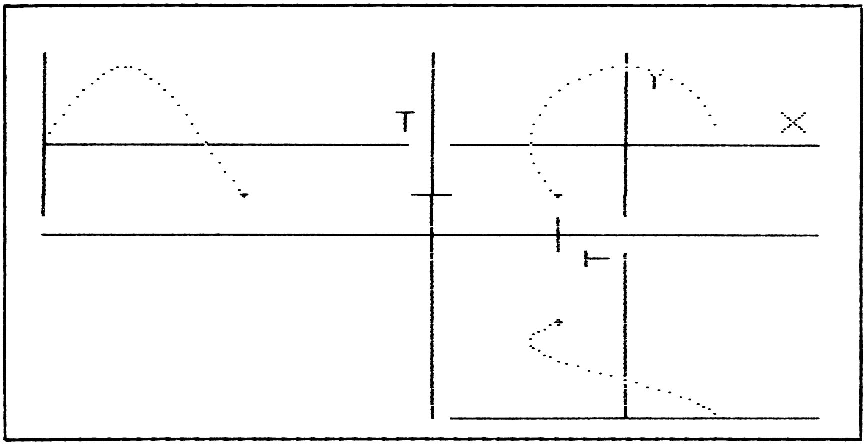
Screen 1. The opening menu is the function menu.

Next, press |G| to go on to the graph menu:

```
|1| X(T)              (to view X(T) in the XT-plane)
|2| Y(T)              (to view Y(T) in the YT-plane)
|3| Y vs X            (to view Y vs. X in the XY-plane)
|4| ALL               (to see all three of the above
                       in motion together.)
|5| CHANGE FUNCTION   (to return to the function menu)
|6| RESCALE T         (to change the T-interval)

|Q|UIT                (to leave the program)
```

Screen 2. The graph menu looks like this (except for the parenthetical explanations).

Press |4| to see the trajectories of X(T), Y(T), and Y vs. X develop as T runs from 0 to 2*PI. If you pause (press any standard key) when T is a little past PI, the display will look something like the one shown in Screen 3.



Screen 3. The circular motion halted at about
T = 1.25 * PI.

Press any key to complete the graphs, and press any key again to return to the graph menu.

You may obtain the graph of X(T) by pressing |1|. Also try the graphs of Y(T) and Y vs. X.

The rescale option #6 on the graph menu lets you enter new values for TMIN and TMAX for the current functions. Use relatively short intervals whenever you can, so that consecutive sample points will lie reasonably close together on the screen.

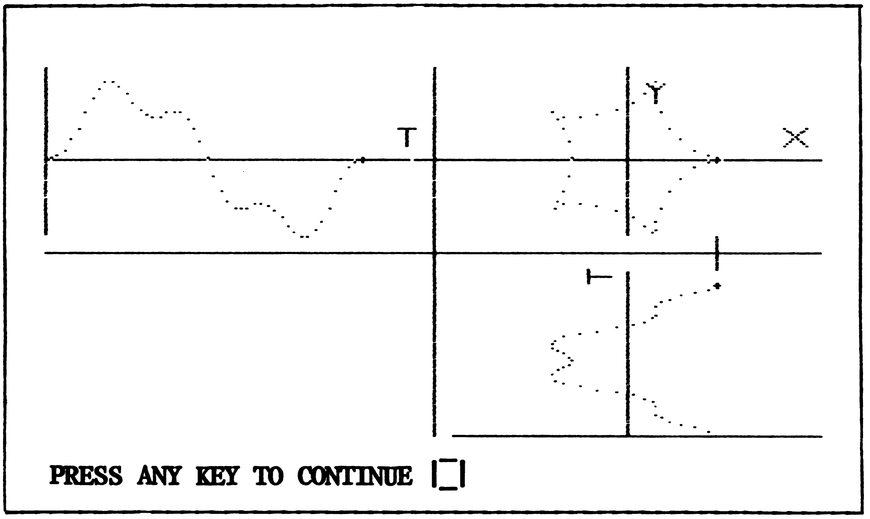
(For an example of what <u>not</u> to do, try the current functions on the ALL Screen with TMIN = 0 and TMAX = 100.)

3. A LOVELY GRAPH

Now return to the function menu and enter

```
X = 4*COS(T) + COS(4*T)        TMIN = 0
Y = 4*SIN(T) - SIN(4*T)        TMAX = 2*PI
```

Then press |G| to go on, and then press |4|. The resulting graph should look like this:



Screen 4. X = 4*COS(T) + COS(4*T),
Y = 4*SIN(T) - SIN(4*T), 0 ≤ T ≤ 2*PI.

4. PROJECTILE MOTION

The vector equation for ideal projectile motion near the surface of the earth is

$$R = \mathbf{i}(v_0 \cos\alpha)t + \mathbf{j}(-\tfrac{1}{2}gt^2 + (v_0 \sin\alpha)t).$$

We can graph this motion by setting

$$X(T) = (v_0 \cos\alpha)*T$$

$$Y(T) = (-\tfrac{1}{2}g)*T*T + (v_0 \sin\alpha)*T.$$

In these equations, v_0 is the projectile's initial velocity, α is the angle of elevation measured from the horizontal, g is the gravitational constant in appropriate units, and T is elapsed time measured from firing at T = 0.

The projectile reaches its maximum height above horizontal ground at

$$T_m = \frac{v_0 \sin\alpha}{g} \text{ seconds}$$

and strikes down range at

$$TMAX = 2T_m \text{ seconds.}$$

If we take $\alpha = \pi/4$ for maximum range, $g = 32$ ft/s^2, and $v_0 = 32\ /2$ ft/s, then

$$v_0\cos\alpha = v_0\sin\alpha = (32\ /2)(1/\ /2) = 32$$

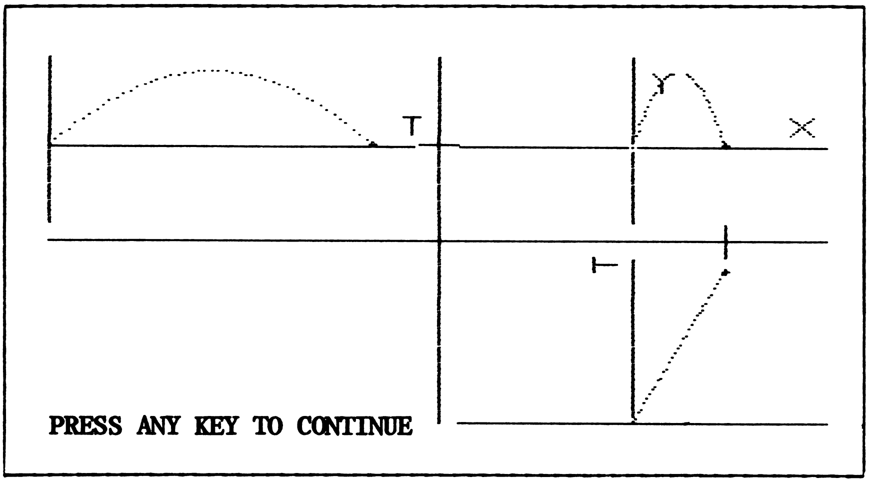
$$TMAX = 2(32/32) = 2.$$

Entering

X(T) = 32*T   Y(T) = -16*T*T + 32*T

TMIN = 0   TMAX = 2

generates the following display on the ALL screen:



Screen 5. The projectile motion Y vs. X, resolved into its X (lower right) and Y (upper left) components.

For a larger picture of Y vs. X, press |3| on the graph menu. Since T is time in this mathematical model, you will see the projectile rise rapidly at first, halt its vertical

motion at maximum height, and fall with increasing velocity back to earth.

PROBLEMS

In Problems 1–10, graph the equations over the given t-intervals, first on the X vs. Y screen, and then on the ALL screen. Enter $4 \cos t$ as 4*COS(T), 2π as 2*PI, and so on.

1. Ellipse: $x = 4 \cos t$, $y = 2 \sin t$, over
(a) $0 \le t \le 2\pi$, (b) $0 \le t \le \pi$, (c) $-\pi/2 \le t \le \pi/2$
2. Hyperbola: $x = \sec t$ $(= 1/\cos t)$,
$y = \tan t$ $(= \sin t/\cos t)$, over the intervals
a) $-1.5 \le t \le 1.5$ b) $-1.4 \le t \le 1.4$
c) $-1 \le t \le 1$ d) $-.5 \le t \le .5$
e) $-.1 \le t \le .1$
3. Parabola: $x = 2t + 3$, $y = t^2 - 1$, $-2 \le t \le 2$
4. Cycloid: $x = t - \sin t$, $y = 1 - \cos t$, over
(a) $0 \le t \le 2\pi$, (b) $0 \le t \le 4\pi$, (c) $\pi \le t \le 3\pi$
5. Trochoid: $x = 2t - \sin t$, $y = 2 - \cos t$ over
(a) $0 \le t \le 2\pi$, (b) $\pi \le t \le 3\pi$, (c) $0 \le t \le 8\pi$
6. Hypocycloid: $x = \cos^3 t$, $y = \sin^3 t$ over
(a) $0 \le t \le 2\pi$, (b) $-\pi/2 \le t \le \pi/2$
7. $x = \cos 2t$, $y = \sin 3t$, $0 \le t \le 4\pi$
8. $x = \cos 3t$, $y = \sin 4t$, over intervals of your choice
9. A nice curve:
$x = 2 \cos t + \cos 2t$, $y = 2 \sin t - \sin 2t$, $0 \le t \le 2\pi$.
What happens when 2 is replaced by -2 in the equations for x and y? Graph the new equations to find out.
10. An even nicer curve:
$x = 3 \cos t + \cos 3t$, $y = 3 \sin t - \sin 3t$, $0 \le t \le 2\pi$.
What happens when 3 is replaced by -3 in the equations for x and y? Graph the new equations to find out.
11. Projectile motion: Graph
$x = (64 \cos \alpha)t$, $y = -16t + (64 \sin \alpha)t$, $0 \le t \le 4 \sin \alpha$,
for the following angles of elevation:
a) $\alpha = \pi/4$ b) $\alpha = \pi/6$ c) $\alpha = \pi/3$
d) $\alpha = 0$ e) $\alpha = \pi/2$ (watch out, here it comes!)

J. Root Finder

1. PURPOSE

This program approximates the roots (zeros) of a continuous function F(X) by the bisection method, the secant method, a modified regula falsi, and a quasi Newton's method.

2. THE METHODS

Each method calculates an initial segment X_0, X_1, ... , X_k of a sequence of real numbers that under favorable circumstances converges to a root of the equation Y = F(X). The convergence depends on the function and method as well as the choice of X_0 or X_0 and X_1. A discussion of convergence may be found in most texts on numerical methods. The algorithms used in ROOT FINDER are the ones described in Werner C. Rheinboldt's article, "Algorithms for Finding Zeros of Functions," The UMAP Journal, Spring 1981, Vol. 2, No. 1, pp. 43-72. You may also enjoy G. H. Gonnet's "On the Structure of Zero Finders," BIT, 1977, No. 17, 170-183.

3. THE EXISTENCE OF ROOTS

To guarantee the existence of a root, we can often apply a corollary of the intermediate value theorem for continuous functions. This corollary says that if a function F(X) is continuous throughout a closed interval

[A,B], and if F(A) and F(B) differ in sign, then the equation F(X) = 0 has at least one solution in the open interval (A,B). Thus the calculation F(−1) = −3 (negative) and F(1) = 3 (positive) reveals that the function $F(X) = 4X^3 - X$ has at least one root between X = −1 and X = 1. In fact, F has roots at X = 0 and X = ±1/2.

4. GRAPH THE FUNCTION FIRST

Once you know that F(X) has a root, your first step should be to graph F to learn roughly where the graph intersects the X-axis and how the graph is shaped.

If the graph of F actually crosses the X-axis (Fig. 1), any of the four root finding methods may be tried. The graph should enable you to choose a suitable starting interval [A,B] containing the root and isolating it from other roots, or, in the case of Newton's method, to determine a suitable starting value X_0.

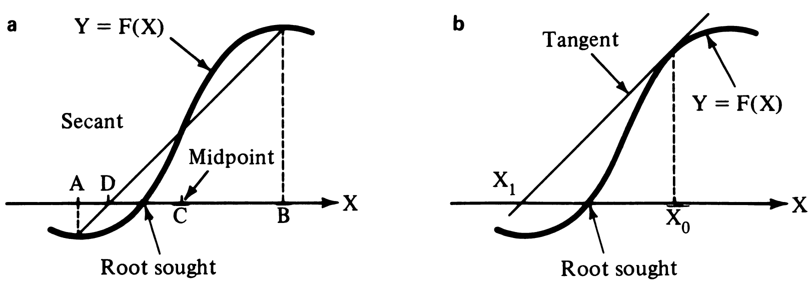


Figure 1. a) With X_0 = A and X_1 = B, the bisection method calculates the midpoint C as X_2. The secant method calculates D as X_2. b) With X_0 as the starting value, Newton's method calculates X_1 to be the point where the tangent to the curve above X_0 crosses the X-axis.

The secant method approximates a root by finding points where secants to the graph near the root cross the X-axis. The modified regula falsi is a variant of the secant method that usually accelerates convergence (Fig. 2).

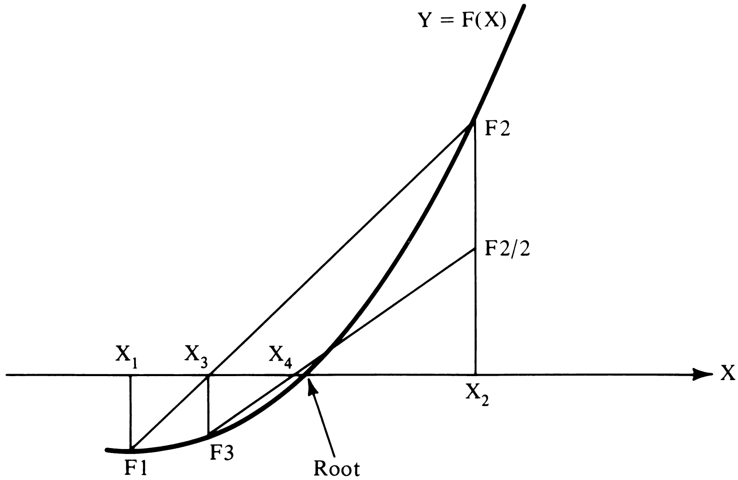


Figure 2. The point X_3 is computed from X_1, F1, X_2, F2 by linear interpolation. Then X_1 is replaced by X_3 (as in bisection, since F2, F3 have opposite sign); but then the value at the fixed end, F2, is replaced by F2/2, so that X_4 is found from X_3, F3, X_2, F2/2. This modification accelerates convergence.

If the graph of F does not cross the X-axis but is instead tangent to it, the program's bisection routine will not apply because it requires F to be negative on one side of the root while positive on the other. The modified regula falsi method will also not apply. The secant method may be tried with proper choices of X_0 and X_1, and in the case of a curve that is convex upward (as in Fig. 3) the secant method will converge to the root if none of the subsequent secant slopes is calculated by the computer to be zero.

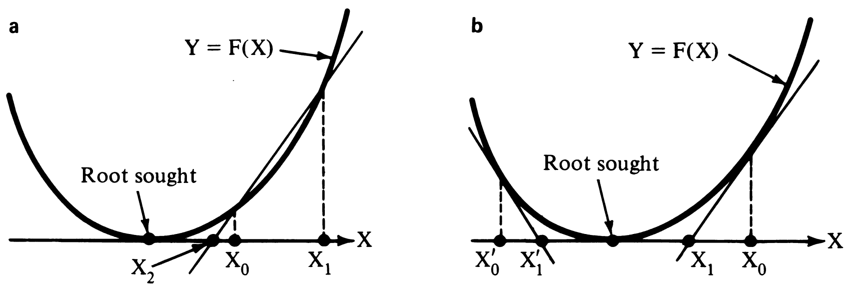


Figure 3. a) If the graph of F looks like this near the root sought, start the secant method by choosing the starting values X_0 and X_1 on the same side of the root. b) Newton's method may be started on either side.

When F is differentiable and its graph does not cross the X-axis at the desired root, try Newton's method. Although the method will fail when a zero derivative is encountered, when it does work it will usually converge faster than the secant method. Use the graph of F to choose a suitable starting value X_0.

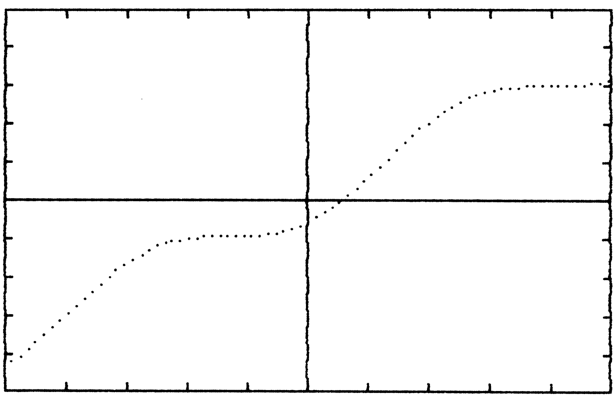
Once you have decided which method to try and have determined the appropriate starting value or values, select the method from the method menu and proceed as in the following examples.

5. STEP BY STEP

Load the program, read the greeting messages, and continue on to the function and domain menu shown in Screen 1. After reading the menu, press |G| to graph F (Screen 2).

```
CURRENT FUNCTION
  F(X) = X - COS(X)

GRAPH REGION
  XMIN = -6
  XMAX = 6
  YMIN = -8
  YMAX = 8

|C|HANGE ENTRY   |R|OOT FINDERS   |G|RAPH   |_|
```

Screen 1. The function and domain menu.



```
-6 < X < 6
-8 < Y < 8
|C|HANGE ENTRY   |R|OOT FINDERS   |_|
```

Screen 2. The graph of F(X) = X - COS(X) shows a root of F between X = 0 and X = 1.

The graph of F(X) = X - COS(X) crosses the X-axis between X = 0 and X = 1. You may use initial X-values in this interval to try all four of the program's root finders.

The messages at the bottom of Screen 2 offer the choice of returning to the function and domain menu to change the function or graphing region (press |C|), or of starting one of the root finding routines. Since the graph in Screen 2 is adequate for localizing the root, i.e., gives adequate information about where to start the approximating sequence X_0, X_1, ... , press |R| to call up the method menu.

```
F(X) = X - COS(X)
-6 < X < 6

    |B|ISECTION METHOD
    |S|ECANT METHOD
    |R|EGULA FALSI METHOD
    |N|EWTON'S METHOD
    |C|HANGE FUNCTION OR GRAPH REGION
    |Q|UIT

PRESS LETTER OF YOUR CHOICE |_|
```

Screen 3. The method menu.

```
BISECTION METHOD
F(X) = X - COS(X)
-6 < X < 6
FIRST GUESSES
X0 = |_|              X1 =
MAXIMUM NUMBER OF ITERATIONS
KMAX = 15
ERROR TOLERANCE
TOL = 1E-03

|RETURN| ACCEPT ENTRY     |ESC| ABORT ENTRY
     ENTRY LIMIT: 15 CHARACTERS
```

Screen 4. Starting the bisection method.

Now press |B| for the bisection method. The display will immediately change to the one shown in Screen 4.

Press four |RETURN|s, pausing between them to watch the screen. The first two will enter the default values $X_0 = 0$ and $X_1 = 1$; the next two will accept the display default values KMAX = 15 and TOL = 1E-03. After the fourth return the message at the bottom of the screen will change to

|C|HANGE VALUES |S|TART SEQUENCE |_|

Press |S| to start the sequence of compuations. The computer will now begin calculating and will display the endpoints of the bisection intervals as it works along. Since KMAX = 15 and TOL = 1E-03, the calculations will halt with X_{15} or when the difference of two consecutive values is less than 1E-03, whichever occurs first. In the present example, the tolerance is reached at step 10, with the results shown in Screen 5.

K	A	B	BISECTION
0	0	1	
1	.5	1	
2	.5	.75	
3	.625	.75	
4	.6875	.75	
5	.71875	.75	
6	.734375	.75	
7	.734375	.7421875	
8	.73828125	.7421875	
9	.73828125	.740234375	
10	.73828125	.739257813	

F(A) = -1.34514983E-03
|C|HANGE ENTRY |R|OOT FINDERS |_|
A AND B LESS THAN 1E-03 APART

Screen 5. The result of applying the bisection method to F(X) = X - COS(X) after specifying $X_0 = 0$, $X_1 = 1$, KMAX = 15, TOL = 1E-03.

The screen display shows that you used the bisection method and that the computation reached the desired tolerance at the tenth iteration. It also shows that the desired root X^* lies between

A = .73828125 and B = .739257813,

the left and right endpoint values of the tenth bisection interval. Rounded to two decimal places, therefore, $X^* = .74$.

As you can see, having achieved a tolerance of 10^{-3} does not mean that you have found the root X^* to three decimal places.

Now press |C| |RETURN| |->| |->| |->| |6| |RETURN| |G| to repeat the computation with the closer tolerance 10^{-6}. The display in Screen 5 will continue from the point it stopped, terminating with

```
15          .739074707        .739105225
F(A) = -1.7449096E-05
|C|HANGE ENTRY    |R|OOT FINDERS    |_|

MAXIMUM NUMBER OF ITERATIONS REACHED
```

This time, the maximum number of iterations is reached before the tolerance is achieved, and the computer tells you so. The display shows the value of F at the last computed left endpoint, and what the left and right endpoints are. The root X^* lies in the interval

$$.739074707 \leq X^* \leq .739105225.$$

Rounded to four decimal places, $X^* = .7391$.

To find the value of X^* more accurately, you can increase the number of iterations. Press |C| to change values. Then press |2| |5| |RETURN| to enter KMAX = 25, press |RETURN| again to accept TOL = 1E-06, and press |G| to continue the computation. The error tolerance is reached at K = 20, with

$$.739084244 \leq X^* \leq .739085197.$$

In the first five decimal places, $X^* = .73908$.

How well does the secant method do with the current parameters? Find out by pressing |R| |S| and |S| again to start the secant computation with $X_0 = 0$, $X_1 = 1$, KMAX = 25, and TOL = 1E-06. The display will show

$$F(.739085112) = -3.59414116E-08$$

after only four iterations. Thus, $X^* \doteq .739085112$. Unlike the successive approximations of the bisection method, however, the successive approximations of the secant method need not bracket X^* and we have no way to tell directly how accurately X^* has been calculated here.

Now try Newton's method. Press |R| and |N|, and then |RETURN| three times to enter $X_0 = 0$ and accept the current values KMAX = 25, TOL = 1E-06. Then press |S| to start the computation. The resulting display, shown here in Screen 6,

K	A	F(A)	NEWTON
0	0	-1	
1	.999999954	.45969761	
2	.75036387	.0189230782	
3	.739112891	4.64561126E-05	
4	.739085133	4.96584107E-10	

F(A) WITHIN 1E-06 OF ZERO.

|C|HANGE ENTRY |R|OOT FINDERS |_|

Screen 6. Newton's method finds X^* to nine places in four steps, but, like the secant method, does not give a way to be sure of the result.

indicates that the value of F at $X_4 = .739085133$ is $F(X_4) = 4.96584107E-10$, a number very close to zero. This suggests that X_4 is very close to X^*, but like the secant display the Newton display gives no interval about X^* from which to tell how close the approximation is.

Return to the bisection method and request $X_0 = 0$, $X_1 = 1$, KMAX = 30, TOL = 1E-08. Tolerance will be reached in twenty-seven steps, the relevant information from the display being

```
27      .73908513       .739085138
F(A) = -4.50017979E-09
F(B) = 7.85439625E-09
A AND B LESS THAN 1E-08 APART
```

Thus, after rounding,

$$X^* = .7390851,$$

to seven places, as the Newton calculation in Screen 6 suggested.

6. PITFALLS OF COMPUTATION

This section gives examples of some of the hazards of numerical root finding.

Example 1. Leaving the domain of the function or its derivative.

Newton's method will not find the root $X^* = 1$ of the function $F(X) = \sqrt{X - 1}$ if the first guess is 4 or greater. With $X_0 = 4$, the method calculates $X_1 = 0$, where F' is not defined (Fig. 4), and the computation stops. With $X_0 > 4$, the method finds $X_1 < 0$, which is not in F's domain. Again, the computation stops.

The secant method encounters a similar problem if $4 \leq X_0 < X_1$.

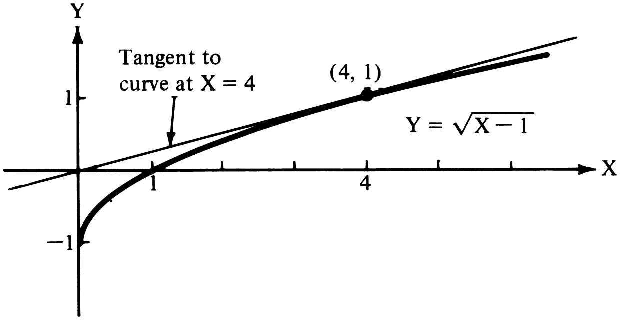


Figure 4. The tangents to the right of X = 4 cross the X-axis at negative X-values.

Example 2. <u>Finding a root different from the one sought</u>.

Figure 4 shows the graph of

$$F(X) = X^4 - X^2 = X*X*(X + 1)*(X - 1).$$

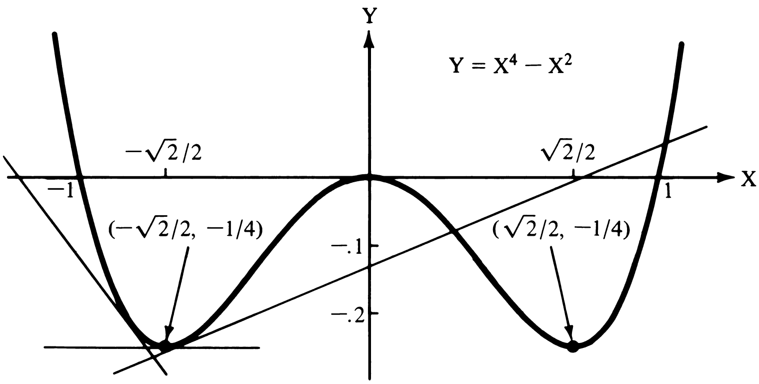


Figure 5. All three roots can be found by starting Newton's method near $-\sqrt{2}/2$.

Newton's method will find the root $X^* = -1$ if X_0 is far enough to the left of $-\sqrt{2}/2$. With X_0 too close to $-\sqrt{2}/2$, however, the computer will encounter a zero slope or a value of X_1 too large to handle. There is a zone just to the right of $-\sqrt{2}/2$ where values of X_0 will lead to

$X^* = 1$ instead of $X^* = 0$.

Likewise, selected values of X_0 near $\sqrt{2}/2$ will lead to $X^* = 1$, $X^* = 0$, $X^* = -1$, or no root at all.

Similar remarks apply to the secant method.

Example 3. Curves that are almost flat near the root.

Some curves are so nearly flat near a root X^* that the secant method encounters a zero slope in the early stages of computation. Try graphing

$$F(X) = (x - 1)\wedge 40, \quad 0 \le X \le 2, \quad -1 \le Y \le 1,$$

and then solving the equation $F(X) = 0$ by the secant method with $X_0 = -1.5$, $X_1 = 2$, KMAX = 5, TOL = 1E-06. The computer will encounter a zero secant slope almost immediately and indicate that it cannot go on. Press |RETURN| to continue, and then press |C| |ESC| and |G| to graph the function.

Other curves are so flat near a root that tolerance is reached before the root is approximated with any useful degree of accuracy. Try the secant method on

$$F(X) = (X - 1)\wedge 21, \quad 0 \le X \le 2$$

with $X_0 = 1.5$, $X_1 = 2$, KMAX = 10, TOL = 1E-06. The computer will stop at K = 2 with the following information:

K	A	B	SECANT
0	1.5	2	

F(A) = 4.76837158E-07
F(B) = 1
F(A) WITHIN 1E-06 OF ZERO

Thus, even with TOL = 1E-06 the estimate of the root is in error by fifty percent.

7. TWO (usually) AVOIDABLE TROUBLES

Two questions to ask if you encounter trouble at the beginning of a computation are:

1. Secant method. Are the function values so nearly equal at X_0 and X_1 that the computer encounters a zero slope right away? For example, the method will work on F(X) = (X - 1)^5 with $X_0 = 0$ and $X_1 = 2$, but not on G(X) = (X - 1)^6.
2. Bisection method. Does the function change sign at the root sought? The algorithm requires it. The method will work on F(X) = (X - 1)^5 with $X_0 = 0$ and $X_1 = 2$, but not on G(X) = (X - 1)^6.

8. PAUSE AND ESCAPE

To pause to inspect a particular value during a computation display, press |CTRL|S| (together). Press |CTRL|S| or |RETURN| to resume.

To escape from a computation and return to the method menu, press |ESC|.

PROBLEMS

In Problems 1-12, estimate the root of F(X) on the given interval with KMAX = 20 and TOL = 1E-6. Compare the results of all four ROOT FINDER methods, or as many as apply. Nine-place answers are given with the problems for comparison.

1. $F(X) = X^2 + X - 1, \quad 0 \leq X \leq 1$
   $X^* = .618033989$
2. $F(X) = X^3 + X - 1, \quad 0 \leq X \leq 1$
   $X^* = .682327803$
3. $F(X) = X^4 + X - 3, \quad 1 \leq X \leq 2$
   $X^* = 1.16403514$

4. $F(X) = X^4 - 2, \quad 1 \leq X \leq 2$
$X^* = 1.18920712$
5. $F(X) = X^3 + 2X - 4, \quad 1 \leq X \leq 2$
$X^* = 1.17950903$
6. $F(X) = X^4 - X^3 - 75, \quad 3 \leq X \leq 4$
$X^* = 3.22857729$
7. $F(X) = X^3 - 3X - 1, \quad -2 \leq X \leq -1$
$X^* = -1.53208889$
8. $F(X) = \sqrt{X} + \sqrt{1 + X} - 4, \quad 0 \leq X \leq 4$
$X^* = 3.515625$
9. $F(X) = 1/(1 - X) + \sqrt{1 + X} - 3.1, \quad .4 \leq X \leq .5$
$X^* = .470194274$
10. $F(X) = \sqrt{1 + X} + SIN(X) - .5, \quad .5 \leq X \leq 0$
$X^* = -.326461807$
11. $F(X) = 2\,COS(X) - \sqrt{1 + X}, \quad 0 \leq X \leq 2$
$X^* = .828360808$
12. $F(X) = X^X - 2, \quad 1 \leq X \leq 2$
$X^* = 1.55961047$
13. $F(X) = \sqrt{2X + 1} - \sqrt{X + 4}$ has a root at $X^* = 3$. Starting with $X_0 = 2$ and $X_1 = 4$, determine how many steps each method takes to find X^*.
14. Find the three zeros of $F(X) = 2^X - X^2$.
15. Starting with $X_0 = .5$ and $X_1 = 2$, how close can you come to the root $X^* = 1$ of $F(X) = (X - 1)^{11}$ with the bisection method?
16. Try finding the root $X^* = 1$ of $F(X) = (X - 1)^{21}$ with the bisection method with $X_0 = 0$, $X_1 = 1.5$, KMAX = 10, and TOL = 1E-03. Will taking TOL = 1E-08 improve the result?
17. Find three solutions of the equation $X^3 = 3X + 1$.
18. Find two solutions of the equation
$X^4 - 2X^3 - X^2 - 2X + 2 = 0$.
19. Find three solutions of the equation $4X^5 - 5X^4 + X = 0$.
20. Find the four solutions of the equation
$2X^4 - 4X^2 + 1 = 0$.

K. Picard's Fixed Point Method

1. PURPOSE

This program enables you to explore the Picard method and use it to solve equations of the form

$$f(x) = x$$

numerically. You may also use the program to estimate the zeros of a function $h(x)$ by applying the Picard method to the equation $f(x) = h(x) + x = x$.

The method may be applied when f is continuously differentiable throughout a neighborhood of the fixed point x and $|f'(x)| < 1$. If $|f'(x)| > 1$, the fixed points of f may be found by applying the method to $g(x) = f^{-1}(x)$ instead.

2. DESCRIPTION

A <u>fixed point</u> of a function f is a value x for which $f(x) = x$. For example, zero is a fixed point of $\sin x$ because $\sin 0 = 0$. Geometrically, a fixed point of f is a point on the x-axis for which the graph of the curve $y = f(x)$ intersects the line $y = x$. (See Fig. 1.)

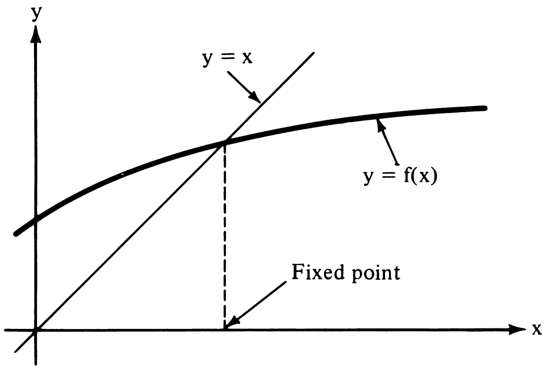


Figure 1. The geometric interpretation of a fixed point.

In this program, fixed points are found by repeated evaluation of f(x). This method, often called Picard iteration, begins with the choice of an x-value, say x_0, and an evaluation of f(x) at this point: $y_0 = f(x_0)$. Next let $x_1 = y_0$ and find $y_1 = f(x_1)$, and then repeat the cycle with $x_2 = y_1$. This procedure determines a sequence of values

$$x_0, x_1, x_2, \ldots$$

satisfying the relation

$$x_n = f(x_{n-1}), \quad n = 1, 2, \ldots .$$

Under favorable conditions, these values will approach a fixed point of f as n increases.

The Picard procedure may be visualized as a path of horizontal and vertical line segments, as shown in Figs. 2 through 5. Start from the point $(x_0,0)$ and move vertically to (x_0,y_0). Then the assignment $x_1 = y_0$ corresponds to a horizontal move to the line y = x, from which the next move is vertical, to (x_1,y_1). The construction is then

repeated: each Picard step after the first is represented by a horizontal segment leading to the line $y = x$ followed by a vertical segment leading to the next computed value.

The Picard method works, for instance, when f and its derivative f' are continuous and $|f'(x)| < 1$ at the fixed point x. For then $|f'| < 1$ throughout an open interval (a,b) containing x, and any choice of x_0 in this interval will lead to x. (The conditions on f here are <u>sufficient</u> conditions, but not always necessary. In some instances the method will find x even if one or more of these conditions fails to hold. See any introductory text on numerical analysis for more about the method and its underlying mathematics.)

Roughly speaking, we can say that if the slope of the curve $y = f(x)$ at the fixed point is strictly between -1 and 1, and if x_0 is near the fixed point, then the path leads to the solution, as in Figs. 2 and 3. If the slope at the fixed point is greater than 1 in absolute value, the path leads away from the solution, as in Figs. 4 and 5.

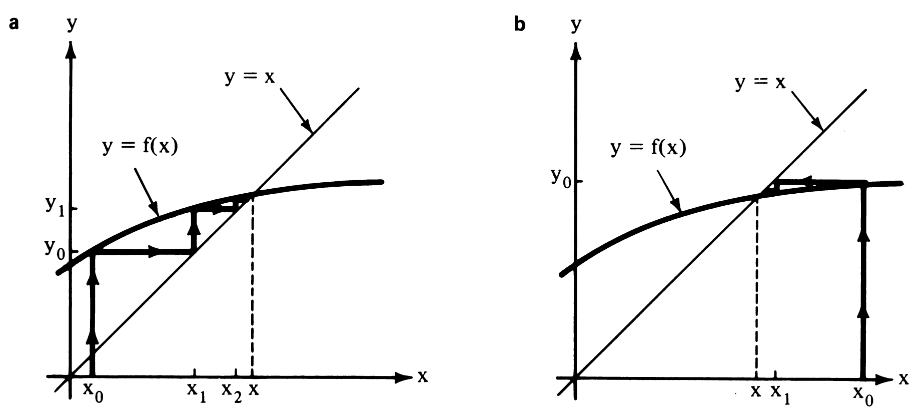


Figure 2. Interpretation of convergence when $0 \leq f'(x) < 1$ at a fixed point x: a) $x_0 < x$; b) $x_0 > x$.

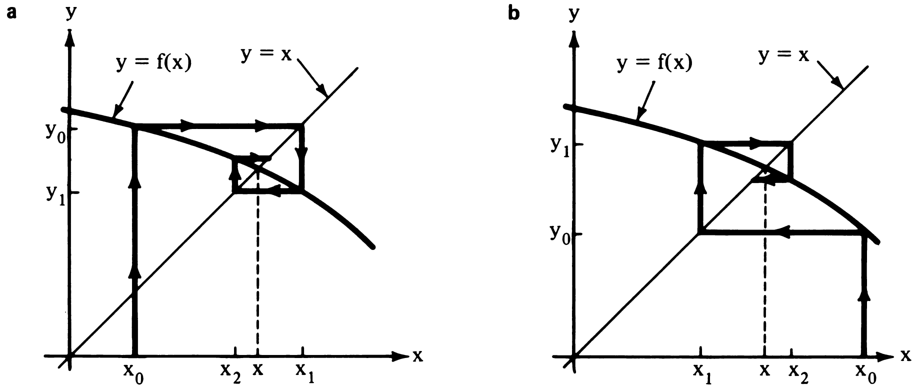


Figure 3. Interpretation of convergence when $-1 < f'(x) \leq 0$ at a fixed point x: a) $x_0 < x$; b) $x_0 > x$.

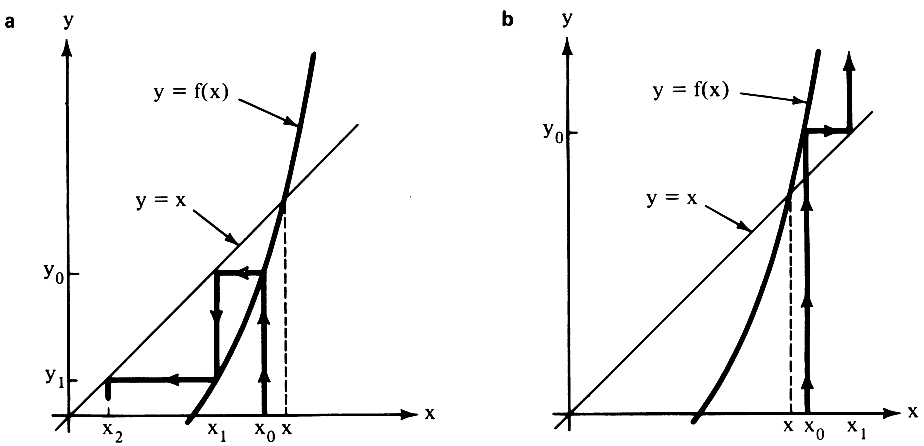


Figure 4. Interpretation of divergence when $f'(x) > 1$ at a fixed point x: a) $x_0 < x$; b) $x_0 > x$.

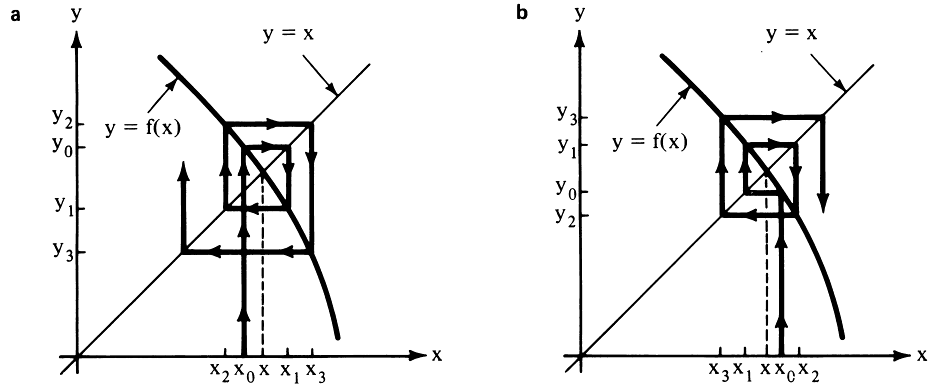


Figure 5. Interpretation of convergence when $f'(x) < -1$ at a fixed point x: a) $x_0 < x$; b) $x_0 > x$.

When $f'(x) = 1$ the procedure may be too slow to be useful. If $f'(x) = -1$, successive values of x_n may straddle the fixed point rather than converge to it, but its value can be approximated by $(x_n + x_{n-1})/2$.

3. WHEN $|f'(x)| > 1$, USE f^{-1}

When $|f'(x)| > 1$ at a fixed point x (so the iteration leads away from x), we can find x by replacing f by its inverse function $f^{-1} = g$.

Since $f(x) = x$ for a fixed point, we have

$$g(x) = g(f(x)) = f^{-1}(f(x)) = x.$$

That is, $g(x) = x$, so that the fixed points of f are to be found among the fixed points of g.

Similarly, if x is a fixed point of g, the equalities

$f(x) = f(g(x)) = f(f^{-1}(x)) = x$ show that x is also a fixed point of f. Thus, f and its inverse g have identical fixed points.

Finally, since $g'(f(x)) \cdot f'(x) = 1$ for differentiable inverse functions whose derivatives are different from zero, the inequality $|f'(x)| > 1$ at a fixed point x implies that $|g'(x)| < 1$. Hence, when $|f'(x)| > 1$, the Picard procedure applied to g leads to a fixed point of f.

4. STEP BY STEP

After loading the program, read the greeting messages. Note that in all computer input and output F(X) denotes the function f(x), called the base function, and G(X) denotes $f^{-1}(x)$. Now continue on to the program menu, shown below, and press |P| to begin Example 1.

```
                 PROGRAM MENU

          |P| .. PROBLEM DISPLAY
          |F| .. GRAPH F(X)
          |G| .. GRAPH G(X)
          |I| .. ITERATE F(X)
          |J| .. ITERATE G(X)
          |Q| .. QUIT

        PRESS LETTER OF YOUR CHOICE  |_|
```

Screen 1. The program menu.

Example 1. Find the fixed points of the function

$$f(x) = x^2.$$

Solution. Press |P| on the program menu, if you have not done so already, to display the problem menu:

```
BASE FUNCTION AND ENDPOINTS

F(X) = X * X
     A = 0              B = 1.1

INVERSE FUNCTION AND ENDPOINTS

G(X) = SQR(X)
     C = 0              D = 1.21

|C|HANGE ENTRY    |G|O ON    |Q|UIT
```

Screen 2. The problem menu.

The base function f(x) is identified in computer notation as

F(X) = X * X

This function was chosen for the first example because it is familiar and its fixed points are known in advance: the roots of the equation $x^2 = x$ are, of course, 0 and 1.

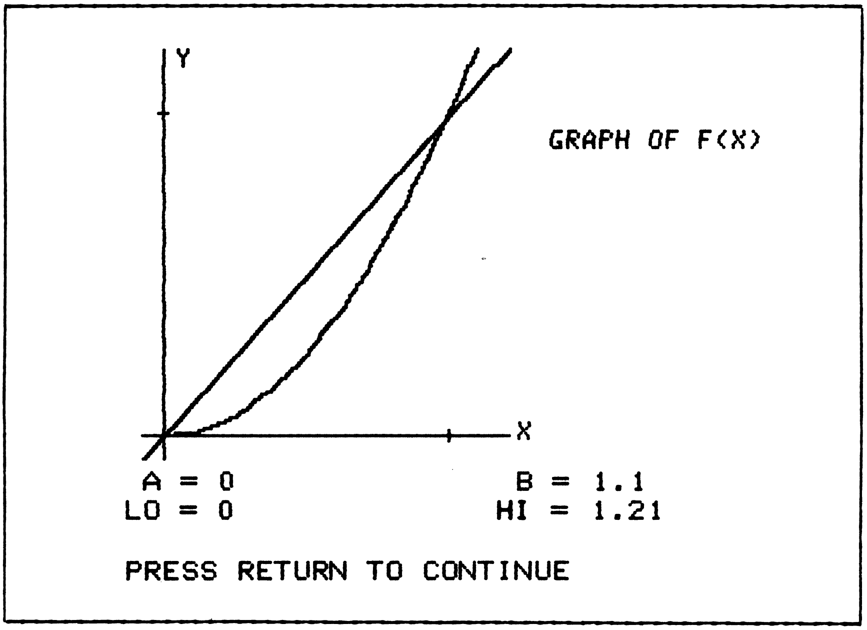
Note the additional information on the screen. The interval for graphing F was chosen to run from 0 to 1.1 to include both fixed points while providing a good screen display. In addition, the inverse function, g(x) = /x, is identified on the screen as

G(X) = SQR(X)

and its plotting interval was chosen to coincide with the range of values taken on by F.

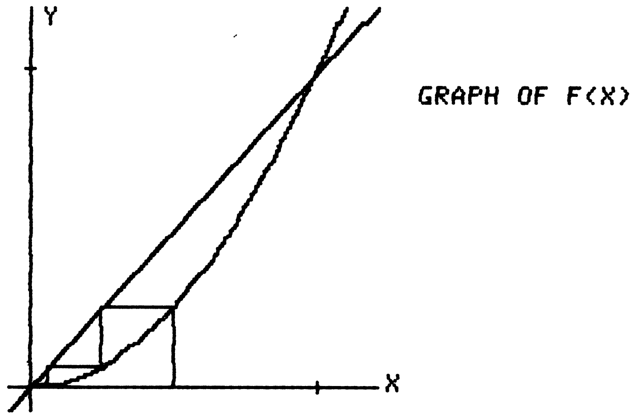
Now press |G| to return to the program menu, then press |F|. A graph of Y = X * X over the interval from 0 to 1.1

will appear quickly on the screen along with a graph of the line Y = X, as shown in Screen 3.



Screen 3. The graph of F(X) = X * X crosses the line Y = X at (0,0) and (1,1). The points X = 0 and X = 1 are fixed points of F.

The graph in Screen 3 shows the two fixed points and indicates that the slope of the curve is less than 1 at x = 0 and greater than 1 at x = 1. Thus, recalling Figs. 2 and 4, we expect a choice of x_0 between 0 and 1 to produce a sequence approaching x = 0. Press |RETURN| for the menu, select |I|, and accept the value .5 for X0 by pressing |G| to go on. The display now shows the geometry of the convergence as well as the final computed function values.



5	1.52587891E-05	2.32830644E-10
6	2.32830644E-10	5.42101087E-20

PRESS RETURN TO CONTINUE

Screen 4. The iteration path from X0 = .5 to X = 0.

The action has been slowed substantially with a time delay routine in the computer program, but it still moves fast. You can halt the program by pressing $\boxed{\text{RETURN}}$, then continue with another $\boxed{\text{RETURN}}$.

When the iteration stops, the numbers at the bottom of the screen show that 6 steps were required to reach the value $5.42101087 \times 10^{-20}$, which approximates the fixed point zero to 19 decimal places. This is better accuracy than is normally expected, since the program stops calculating as soon as the difference between two successive values of the function is less than 10^{-8} in absolute value.

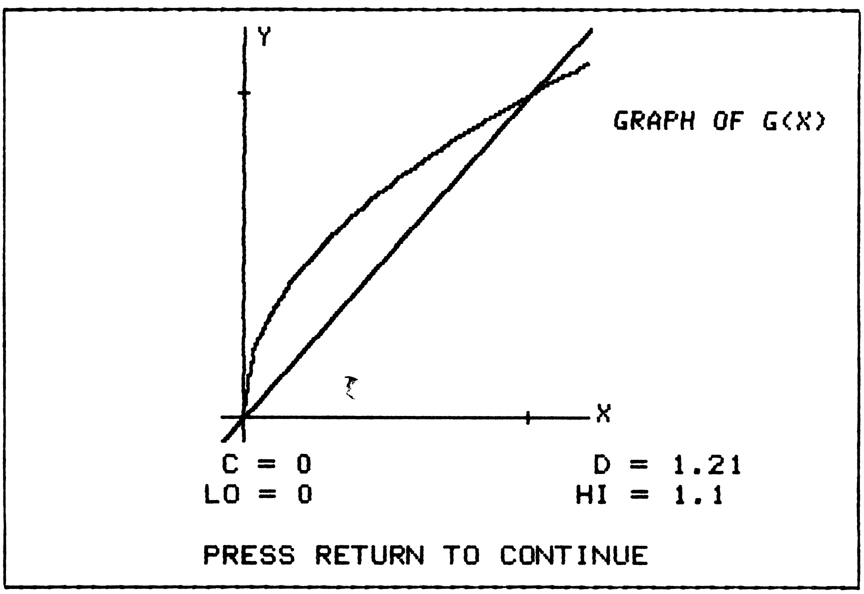
Press $\boxed{\text{RETURN}}$ to see the entire table of values. The display shows the number of steps, the starting X-value .5, and the computed values .25, .0625, ...

Now press |RETURN| to return to the program menu, then |P| to review the problem. Note especially the plotting interval for the inverse function. Below C and D the display now shows the minimum and maximum values that were computed for F in plotting Y = F(X):

LO = 0 HI = 1.21

Although these values are not needed for the present problem, their automatic appearance will be useful in problems that require entry of these values for plotting Y = G(X).

Press |G| to go on, and |G| on the program menu. The graph of G(X) = SQR(X) for X from 0 to 1.21 appears after computation of a table of values:



Screen 5. The fixed points of G(X) = SQR(X) are also X = 0 and X = 1.

Again the fixed points 0 and 1 are apparent, but this time the slope of the curve at x = 1 lies between 0 and 1. The iteration can start anywhere in the interval (A,B).

Press |RETURN|, and then |J| when the program menu appears. An opportunity to change the value of X0 is presented, but X0 = .5 will do, so press |G| to start the iteration. This time the function values approach 1, as desired, but the convergence is slow. At step 27 the stopping criterion is finally satisfied, and the fixed point 1 is approximated by .999999995, which is accurate to 8 decimal places since the error is 5×10^{-9}.

Now press |RETURN| to see the computed values. Since the screen holds only 24 lines of text, the first few values have scrolled off the screen. Press |RETURN|, |J|, and |G| to repeat the full display of values. The action can be stopped any time with one |RETURN|, then continued with another. This time you should get a better view of the values

$$.5, \quad .707106781, \quad .840896415, \quad \ldots,$$

that are calculated as the iteration proceeds.

5. THE F-G TOGGLE SWITCH FEATURE

In connection with Example 1, we point out an additional feature of the program. From the program menu press |F| to graph Y = F(X), and then |RETURN| and |G| to graph Y = G(X). Keys |F| and |G| can now be used as a toggle switch to alternate between the two graphs without intermediate returns to the menu. This action provides a vivid display of the symmetry in the line $y = x$ that characterizes the geometric relationship between a function and its inverse. Try it.

6. ANOTHER NICE FEATURE: PRESS |ESC| TO SKIP G IN THE PROBLEM DISPLAY

Example 2. Find the fixed point of the function

$$f(x) = 1 + x/2$$

using Picard iteration with $x_0 = 1$.

Solution. Again, the example is chosen to confirm a solution we know in advance: the equation $1 + x/2 = x$ has one root, $x = 2$. Also known in advance is that the graph of f is a straight line with slope 1/2. Since the slope is less than one in absolute value, the Picard procedure will succeed with f. The inverse function is not needed.

Press |P| on the program menu, then |C|, and enter

F(X) = X/2 + 1   A = 0   B = 3

When the value of B is entered, the cursor will jump to the formula for G(x). Since the inverse function is not needed in this problem, a direct exit from the change mode can be made by pressing |ESC|. Now press |G| and |F| to graph F. Then return to the program menu, press |I|, enter X0 = 1, and press |G| to start the iteration. In 27 steps the procedure terminates with $x = 1.99999999$, which approximates the fixed point 2 with an error of 10^{-8}. Press |RETURN| to see steps 6 through 27.

To repeat the display of values from the beginning, press |RETURN|, then |I|, then |G|, stopping and continuing as desired by pressing |RETURN|.

To experiment, try other values of X0, say -10, -20, and 50. If you wish to see the action graphically, you will need to construct the graph of F each time before starting the iteration.

It is also instructive to enter the inverse,

G(X) = 2 * (X - 1)

Take C = 1, D = 2.5, and compare the graphs of F and G. Note what happens when the Picard iteration is applied to G.

Example 3. Find the fixed points of the function

$$f(x) = x^3 + 1.$$

Solution. The problem is to find all points x for which the graph of the curve $y = x^3 + 1$ intersects the line $y = x$. This geometric interpretation is especially useful in this problem since the given curve is simply the cubic $y = x^3$

translated up 1 unit. With this in mind it is not hard to imagine a fixed point somewhere between −2 and −1. This can be confirmed by evaluating the function at the points $x = -2$ and $x = -1$:

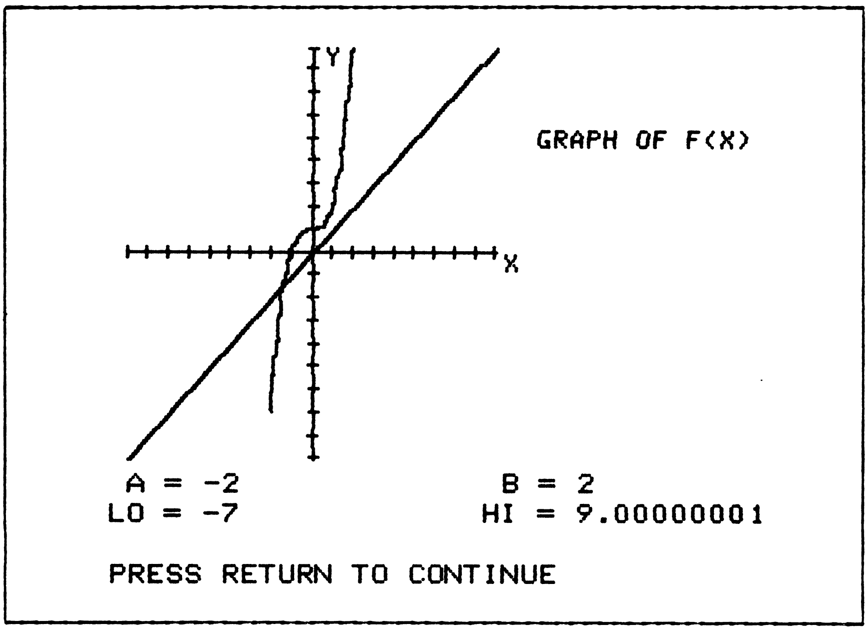
$$f(-2) = -7, \quad \text{and} \quad f(-1) = 0.$$

Since $f(-2) < -2$ and $f(-1) > -1$ the graph of $y = f(x)$ moves from a point below the line $y = x$ to a point above it as x varies from −2 to −1, and since f is continuous throughout the interval, there must be a point of intersection.

Select |P| on the program menu, press |C|, enter

F(X) = X*X*X + 1, A = −2, B = 2,

and press |ESC| to complete the entry. If the entries are correct, press |G| and then |F| to graph F.



Screen 6. The fixed point of F(X) = X*X*X + 1 lies between X = −2 and X = −1.

The display in Screen 6 confirms the existence of a fixed point between −2 and −1, and indicates that this should be the only one. However, the display also shows that the Picard procedure must be applied to the inverse function since the slope of the curve is greater than 1 at the fixed point.

Press |RETURN|, then |P|, and enter

G(X) = −((1 − X)∧(1/3)) C = −2 D = 0

When the entries have been checked, press |G| to go on, then |G| to graph G. The display shows the fixed point and indicates that the iteration will succeed if X0 lies between −2 and 0. Press |RETURN| and |J| and enter 1 for X0. Then press |G| to begin the computation. The value −1.32471796, which approximates the fixed point to 8 places, is reached at step 12.

Now press |RETURN| to display the computations. When you are ready, press |RETURN| again to prepare for the next example.

Example 4. Find the fixed point of the function

$$f(x) = (2x^3 - 1)/(3x^2 - 1)$$

that lies between −2 and −1.

Solution. Press |P| on the problem menu and enter

F(X) = (2*X*X*X − 1)/(3*X*X − 1),

A = −2, and B = −1. Press |ESC|, then |G| when the entries are checked. Graph F to confirm the existence of a fixed point in the interval. Then return to the problem menu, press |I|, enter X0 = −1, and press |G| to start the iteration. The fixed point approximation, −1.32471796 to 8 places, is reached at step 6.

Had we been unable to see by inspection that F has a fixed point between −2 and −1, it would first have been *necessary* to conduct an independent search to localize the fixed point and determine suitable values for A and B.

7. CONNECTION WITH NEWTON'S METHOD

The functions in Examples 3 and 4 have the same fixed point even though they are quite different. While such behavior is often coincidental, in this case there is a good reason.

First note that the equation

$$x^3 + 1 = x$$

is equivalent to

$$x^3 - x + 1 = 0.$$

If Newton's method is applied to

$$h(x) = x^3 - x + 1,$$

the algorithm

$$x_{n+1} = x_n - h(x_n)/h'(x_n)$$

generates the same sequence we would get by applying the Picard method to the function

$$x - h(x)/h'(x)$$

(for any choice of x_0 near the root). The function in Example 4 was originally obtained by substituting the formula for h(x) into this expression and simplifying the result:

$$\begin{aligned} x - h(x)/h'(x) &= x - (x^3 - x + 1)/(3x^2 - 1) \\ &= (2x^3 - 1)/(3x^2 - 1) \end{aligned}$$

The iteration sequence in Example 4, which was based on Newton's method, requires only 6 steps for the same accuracy that required 12 steps in Example 3. This behavior is typical: Newton's method is generally much faster than Picard's method. The price paid for the gain in the convergence rate, however, is that of having to find the derivative when Newton's method is applied.

8. BE SURE TO HAVE THE REAL INVERSE

Example 5. Find the solution of the equation

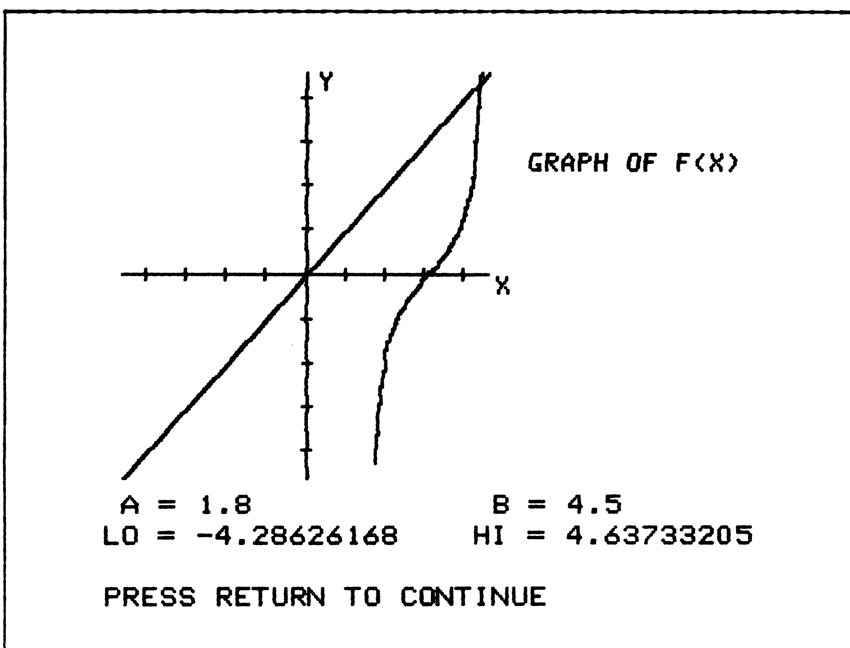
$$\tan x = x$$

that lies between $\pi/2$ and $3\pi/2$.

Solution. This example underscores the need for an understanding of inverse functions. Press |P| on the program menu, then |C|, and enter

F(X) = TAN(X) A = 1.8 B = 4.5

Then press |ESC| and graph F after checking the entries. (The values 1.8 and 4.5 were chosen to provide a good display while avoiding $\pi/2$ and $3\pi/2$: because of discontinuities, the inclusion of either value in the plotting interval would result in an overflow error, an illegal quantity error, or an artifact in the display since the computer is programmed to connect the points it plots.)



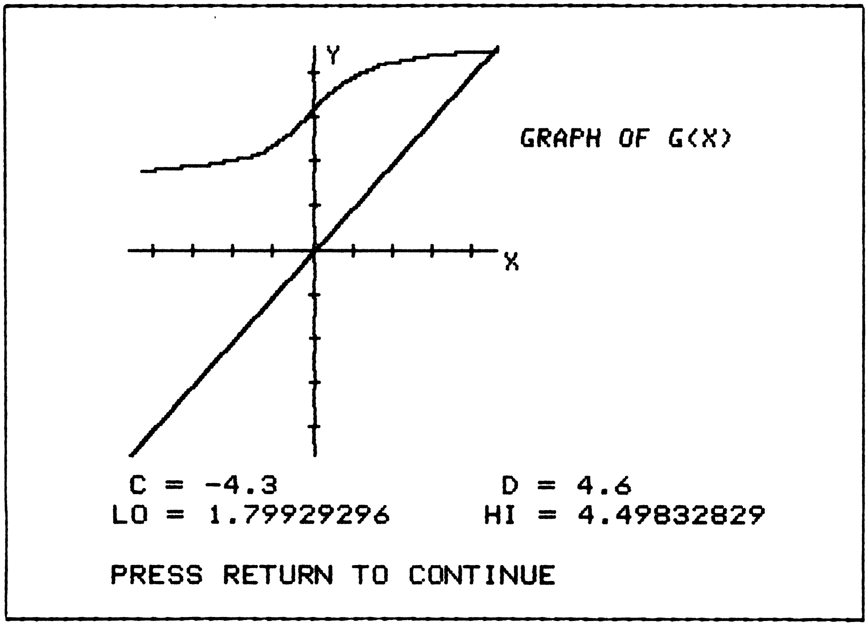
Screen 7. The graph of F(X) = TAN(X), 1.8 ≤ X ≤ 4.5.

The graph in Screen 7 shows a fixed point near 4.4 but also shows that the slope of the curve exceeds 1 at that point, so the iteration procedure must be applied to the inverse function.

Now go back to the program menu, press |P|, and then |C| to enter G(X). Special care is needed here. It is tempting to use $\arctan x$ as the inverse of $f(x)$, but $\arctan x$ is the inverse of the function $\tan x$ restricted to the interval $-\pi/2 < x < \pi/2$. The inverse of $\tan x$ over the interval $\pi/2 < x < 3\pi/2$ is $\pi + \arctan x$, so enter

G(X) = PI + ATN(X)

Take the values of C and D to be the LO and HI values displayed in Screen 7, rounded as C = –4.3 and D = 4.6. Then graph G, and take a moment to toggle between the graphs of F and G.



Screen 8. The graph of G(X) = PI + ATN(X).

Return to the program menu and find the fixed point of F by iterating G. With X0 = 0, the iteration terminates at step 9 with X = 4.49340945.

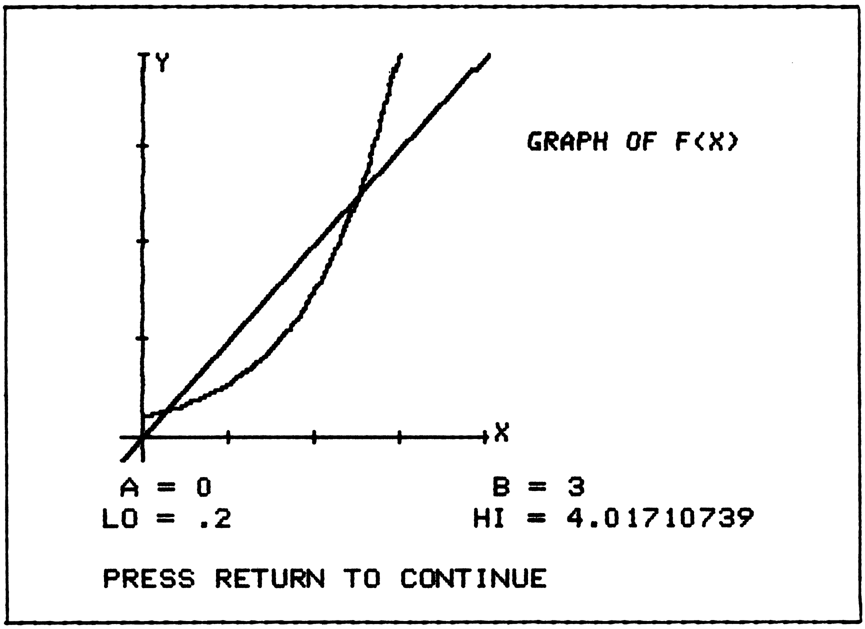
Example 6. Find the fixed points of the function

$$f(x) = e^x/5.$$

Solution. The graph of

$$F(X) = EXP(X)/5,$$

over the interval $0 \leq X \leq 3$ in Screen 9 reveals fixed points near .2 and 2.5.



Screen 9. The function EXP(X)/5 has two fixed points.

The graph in Screen 9 also shows that the inverse function will be needed to find the right-hand fixed point, where the slope of F exceeds 1. Press |RETURN|, then |P| and |C|, enter

G(X) = LOG(5 * X)   C = .2   D = 4.0171

and graph G. Redraw the graph of F(X), and use the F-G toggle to confirm the inverse function relationship.

Iteration on F(X) with X0 = 0 returns the fixed point .259171101 at step 14. Iteration on G(X) with X0 = 2 returns the fixed point 2.54264135 at step 20.

PROBLEMS

In Problems 1-14, first graph $f(x)$. Where needed, also graph the inverse function $g(x)$, and check the inverse relationship graphically. Find all fixed points of f by iterating f or g, using values of x_0 determined after considering the graphs.

1. $f(x) = x/4 + 1$
2. $f(x) = 2x/3 + 1$
3. $f(x) = 3x + 1$
4. $f(x) = \sqrt{x + 1}$
5. $f(x) = \cos x$
6. $f(x) = 1/x^2$
7. $f(x) = 1/(1 + x^2)$
8. $f(x) = 1/(1 + x)$
9. $f(x) = (x^2 + 1)/(2x - 1)$
10. $f(x) = 1 + x^{1/3}$
11. $f(x) = 1 + \sqrt{x - 0.8}$
12. $f(x) = 2 + \arctan 2x$
13. $f(x) = \arctan(2 + x)$
14. $f(x) = 1 + (1/2)\sin x$
15. $f(x) =$ Let $f_1(x) = x^3 - x^2 + .5$
    a) Graph $f_1(x)$ on the interval from -1 to 1.5.
    b) Iterate $f_1(x)$ with $x_0 = 0$ to obtain the middle of the three fixed points.
    c) Iterate $f_2(x) = (x - .5)/(x^2 - x)$ with $x_0 = -0.8$ to obtain the one on the left.
    d) Iterate $f_3(x) = (x^2 + x - .5)^{1/3}$ with $x_0 = 1$ to obtain the one on the right.
16. a) What happens in Example 1 when either iteration is started with X0 = 0 or X0 = 1?
    b) What happens when the iteration of F is started with an X0 for which |X0| < 1, or an X0 for which |X0| > 1? To find out, try X0 = -.5, -2, and 2.

L. Integration

1. PURPOSE

This program enables you to watch rectangular approximations fill the region between the graph of a function Y = F(X) and an interval on the X-axis as the corresponding Riemann sums approach the value of the integral of F over the interval.

2. DESCRIPTION

You enter the function, the interval's endpoint values, and the number of rectangles. The program graphs the function and calculates two Riemann sums: one obtained by using the smaller endpoint value of the function on each subinterval, the other by using the larger endpoint values. You can toggle back and forth between the graphs of the two sets of rectangles. For each subdivision, the trapezoidal sum is given for comparison.

3. STEP BY STEP

Load the program, read the greeting messages, and move on to the input menu shown in Screen 1.

```
INTEGRAND AND LIMITS
  F(X) = COS(X)

  XMIN = 1.57079633
  XMAX = 3.14159265

NUMBER OF SUBINTERVALS
  N = 20

|C|HANGE ENTRY    |G|O ON    |Q|UIT    |_|
```

Screen 1. The input menu.

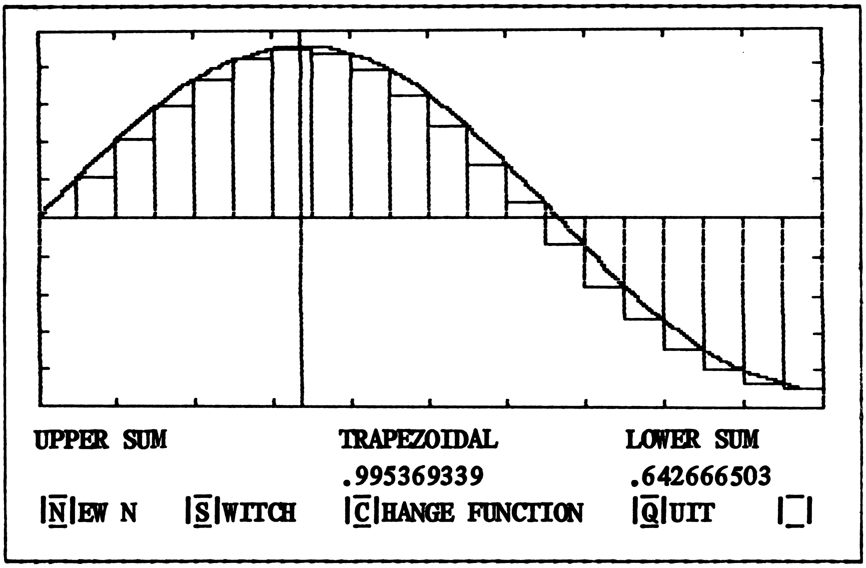
Accept the current function and the current domain endpoint values by pressing |G|. The message "PLEASE BE PATIENT..." will appear as the computer begins to calculate the table of function values for the ensuing graphics display, shown in Screen 2.

Each rectangle in the approximation shown in Screen 2 is obtained by evaluating the function at the endpoints of the subdivision interval that forms the base of the rectangle and choosing the lower of the two function values to determine the rectangle's height. The associated Riemann sum is called the lower endpoint sum for the subdivision.

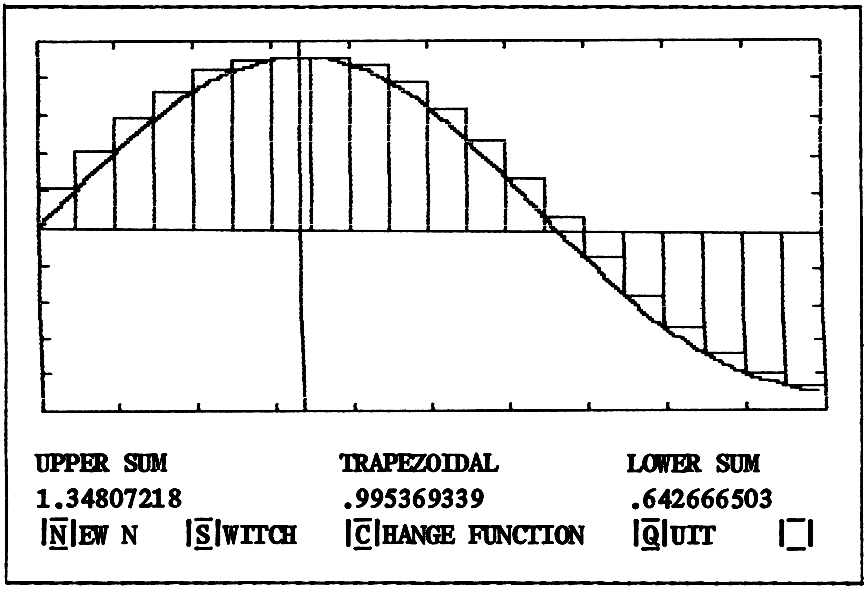
As Screen 2 shows, the lower endpoint sum in this case is .642666503. For comparison, the trapezoidal approximation sum for N = 20 is .995369339. The exact value of the integral, by the way, is

$$\int_{-\pi/2}^{\pi} \cos x \, dx = \sin x\Big]_{-\pi/2}^{\pi} = 0 - (-1) = 1.$$

Now press |S| to see the <u>upper endpoint sum</u>, which uses the upper endpoint function value in each subinterval to determine each rectangle's height, as shown in Screen 3. The trapezoidal and lower endpoint sums are shown for comparison.



Screen 2. Lower endpoint sum with N = 20 rectangles.



Screen 3. The upper endpoint sum with N = 20 rectangles is 1.34807218.

Now that both sets of rectangles (upper and lower) have been drawn, you may toggle between them by pressing |S|. Try it.

Here are the results for seven different values of N:

N	UPPER	TRAPEZOID	LOWER
2	3.3321622	.487983861	-2.35619448
4	2.55904212	.881573567	- .795894986
10	1.68248225	.981425639	.280369027
20	1.34807218	.995369339	.642666503
50	1.14058484	.999259677	.857934551
100	1.07049496	.999814938	.929134918
280	1.02522108	.999976406	.974731729

Two hundred eighty is the limit of the screen's resolution and the largest value of N the program will accept.

The virtue of this program is its ability to calculate Riemann sums and display rectangular approximations simultaneously. As a calculator alone, the program is relatively slow. It took more than half a minute to calculate each of the upper and lower sums for N = 280 in the table above, and they agree with the exact value of the integral to only two digits when rounded. Using Simpson's rule in the program INTEGRAL EVALUATOR produced the following table in four seconds:

S2 = 1.43604331
S4 = 1.01277014
S10 = 1.00028138
S20 = 1.00001724
S50 = 1.00000044

The sum S50 compares very favorably with the exact value of 1 for the integral.

PROBLEMS

Graph the upper and lower endpoint sums for the following integrals for N = 4, 10, 20, and 50.

1. $\int_1^2 \frac{1}{x}\,dx = \ln 2$

2. $\int_{-\pi}^{\pi} \cos x\,dx = 0$

3. $\int_0^1 (x^2 + 1)dx = \frac{4}{3}$

4. $\int_0^1 (x - 1)dx = \frac{1}{2}$

5. $\int_0^{\pi/4} \sec^2 x\,dx = 1$

6. $\int_{-1}^1 |x|dx = 1$

7. $\int_{-1}^1 \sqrt{1 - x^2}\,dx = \frac{\pi}{2}$

8. $\int_{-2}^2 \sqrt{1 - (x/4)^2}\,dx = \pi$

9. $\int_0^{2\pi} \frac{1}{2}\,[\sin x + |\sin x|]\,dx = 2$

M. Integral Evaluator

1. DESCRIPTION

This program evaluates integrals of the form

$$\int_b^a f(x)\,dx$$

by the trapezoidal rule, Simpson's rule, and Romberg integration, and enables you to compare the three results. You key in a formula for f(x), the values of a and b, the number of subintervals for the trapezoidal and Simpson's rules, and an error tolerance for the Romberg integration. Special care is required if f has a removable discontinuity at a or b. For such functions you must also supply the values of f(a) and/or f(b) that make f continuous. When this is not possible, you may still be able to transform the given integral into one to which the program applies.

2. THE NUMERICAL METHODS

The trapezoidal rule and Simpson's rule are often introduced in calculus with the definite integral. Under the trapezoidal rule, for any choice of a positive integer n, the

interval [a,b] is partitioned into n subintervals of equal length $h = (b - a)/n$ by the points

$$x_i = a + ih, \; i = 0, 1, \ldots, n.$$

The ordinates of the curve $y = f(x)$ above these points are then given by

$$y_i = f(x_i).$$

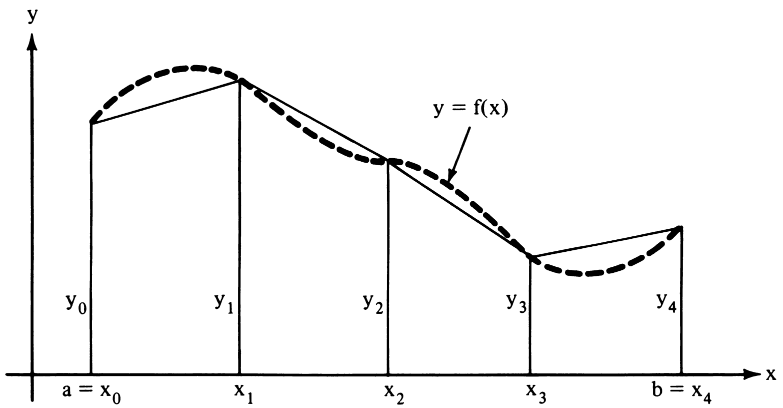


Figure 1. Trapezoidal rule: The area under the curve $y = f(x)$ is approximated by summing areas of trapezoids.

Figure 1 shows the case $n = 4$. In general, the nth trapezoidal approximation to the integral is given by

$$T_n = \frac{h}{2}(y_0 + 2y_1 + 2y_2 + \ldots + 2y_{n-1} + y_n).$$

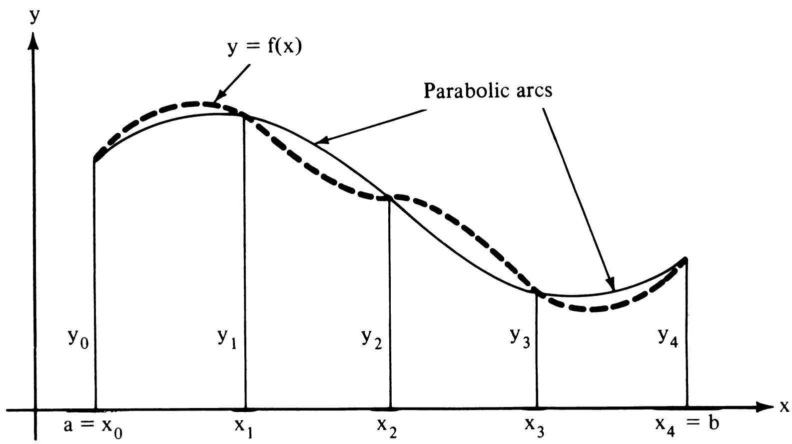


Figure 2. Simpson's rule: The area under the curve y = f(x) is approximated by summing areas under parabolic arcs.

For Simpson's rule, n must be an <u>even</u> positive integer (Fig. 2), and the nth approximation to the integral is given by

$$S_n = \frac{h}{3}(y_0 + 4y_1 + 2y_2 + 4y_3 + \ldots + 2y_{n-2} + 4y_{n-1} + y_n).$$

No justification of either method will be given here. We note only that the trapezoidal error is given by

$$\int_a^b f(x)dx - T_n = -(b - a)h^2 f''(c)/12,$$

where $a < c < b$, and the Simpson error by

$$\int_a^b f(x)dx - S_n = -(b - a)h^4 f^{(iv)}(c)/180$$

where, again, c is some point in the interval (a,b).

Romberg integration comes from representing the error in the nth trapezoidal approximation as a series in powers of $1/n^2$:

$$\int_a^b f(x)dx - T_n = \frac{A_1}{n^2} + \frac{A_2}{n^4} + \frac{A_3}{n^6} + \ldots, \qquad (1)$$

where the A_i are constants that depend on a, b, and f but are independent of n. The key idea is then to eliminate the first term in the series to get an approximation whose error decreases as $1/n^4$ instead of $1/n^2$. To accomplish this, we first substitute 2n for n in Eq.(1) to obtain

$$\int_a^b f(x)dx - T_{2n} = \frac{A_1}{4n^2} + \frac{A_2}{16n^4} + \frac{A_3}{64n^6} + \ldots,$$

then combine the errors for n and 2n algebraically to obtain

$$4T_{2n} - T_n = 3\int_a^b f(x)dx + \frac{3A_2}{4n^4} + \frac{15A_3}{16n^6} + \ldots .$$

This equation suggests that if the quantity

$$T'_n = (4T_{2n} - T_n)/3$$

is used to approximate the integral, then accuracy may be improved, since the resulting error is of the order $1/n^4$.

The procedure by which we constructed T'_n, which is called an extrapolation, may be iterated as follows: form

$$T'_{2n} = (4T_{4n} - T_{2n})/3,$$

then extrapolate again to form

$$T''_n = (16T'_{2n} - T'_n)/15,$$

and continue the process to form the triangular array

$$\begin{array}{llll} T_1 & & & \\ T_2 & T_1' & & \\ T_4 & T_2' & T_1'' & \\ T_8 & T_4' & T_2'' & T_1''' \end{array}$$

The values along the nth row (counting from n = 0) can be found by the formula

$$T^{(i)}_{2^{n-1}} = \left(4^i T^{(i-1)}_{2^{n-i-1}} - T^{(i-1)}_{2^{n-1}}\right) \Big/ (4^i - 1)$$

for i = 1, 2, . . ., n, where the notation

$$T^{(0)}_{2^n}$$

is interpreted to mean the trapezoidal approximation T sub 2^n.

The last value $T_1^{(n)}$ on the nth row is denoted by R_n. The program INTEGRAL EVALUATOR displays the values R_n as the computation proceeds; the output appears on the screen in the form

R0 = T_1

R1 = T_1'

R2 = T_1'',

. . . ,

with the final approximation denoted by R.

The Romberg procedure is normally terminated after the calculation of a preset number of rows or when a prescribed error criterion is satisfied. INTEGRAL EVALUATOR uses a combination of these conditions: for a user-chosen value of ε (whose computer variable name is E) the program terminates when the condition $|R_n - R_{n-1}| < \varepsilon |R_n|$ is satisfied for any value of $n > 2$, or when the tenth row (with n = 9) is computed, whichever comes first.

You can find more about Romberg integration in Methods of Numerical Integration, by P. J. Davis and P. Rabinowitz (New York: Academic Press, 1975).

3. STEP BY STEP

Load the program from the disk menu, read the greeting message, and continue to the program menu shown in Screen 1. The examples all start from this menu.

```
                 PROGRAM MENU
           CHOOSE TYPE OF INTEGRAND

  |C| .. F(X) CONTINUOUS
  |D| .. F(X) DISCONTINUOUS AT A OR B
           OR
  |Q| .. QUIT

   PRESS LETTER OF YOUR CHOICE |_|
```

Screen 1. The program menu.

Example 1. Evaluate

$$\int_0^1 (1 + x^2)dx$$

by the program's three methods, and compare the results. Use n = 2, 5, 10, 20, 50 for the trapezoidal rule; n = 2, 4, 10, 20, 50 for Simpson's rule, and $\varepsilon = 0.00001$ for the Romberg integration.

Solution. Since the integrand $f(x) = 1 + x^2$ is continuous, press |C| on the program menu. When the integration method

menu appears (Screen 2), press |T| to select the trapezoidal rule. The display will change to the one shown in Screen 3.

```
          TYPE OF INTEGRAND
             F(X) CONTINUOUS

                 CHOOSE INTEGRATION METHOD
         |T| .. TRAPEZOIDAL RULE
         |S| .. SIMPSON'S RULE
         |R| .. ROMBERG INTEGRATION
                  OR
         |M| .. PROGRAM MENU

          PRESS LETTER OF YOUR CHOICE |_|
```

Screen 2. The integration method menu.

```
   INTEGRAND AND LIMITS OF INTEGRATION
     F(X) = 1 + X * X

        A = 0              B = 1

   TRAPEZOIDAL APPROXIMATIONS TO BE FOUND
   T2      T5      T10      T20      T50

   |C|HANGE ENTRY  |G|O ON   |M|ENU  |Q|UIT   |_|
```

Screen 3. The initial trapezoidal method display.

Since the default problem in the program is the correct one for this example, press |G| to execute the integration routine. The approximations are then computed and displayed:

T2 = 1.375
T5 = 1.34
T10 = 1.335
T20 = 1.33375
T50 = 1.3334

Three-place accuracy is obtained with 50 subintervals (the exact value of the integral is 4/3).

Now press |RETURN| to return to the method menu. The menu now includes the option

|H| .. HARD COPY OF CURRENT RESULTS.

If your computer is connected to a printer and you know the slot number of the interface card, you may obtain a copy of the calculated values by pressing |H|, then the slot number, then a |RETURN| to confirm your choice.

To continue the demonstration, press |S| for Simpson's rule. The problem display will be similar to the one in Screen 3, except with T replaced by S, and the second approximation with $n = 5$ replaced by $n = 4$. (For Simpson's rule, n must be even.) Press |G| for the integration. Each of the computed values is 1.33333333, since Simpson's rule returns the true value of the integral of a quadratic function for all values of n.

Now continue with a |RETURN|. The trapezoidal values computed earlier will be reproduced below the Simpson's rule values for comparison.

Continue by pressing |RETURN| and then |R| for the Romberg display shown in Screen 4.

```
INTEGRAND AND LIMITS OF INTEGRATION
  F(X) = 1 + X * X
                    A = 0          B = 1

ERROR TOLERANCE
     E = 1E-05

|C|HANGE ENTRY  |G|O ON   |M|ENU  |Q|UIT  |_|
```

Screen 4. The initial Romberg display.

Since the default value of ε (shown in Screen 4 as E = 1E-05, which represents 10^{-5}) is appropriate for this problem, press |G| for the integration. The resulting display shows

R0 = 1.5
R1 = 1.33333333
R2 = 1.33333333

R = 1.33333333.

The computed values are consistent with several facts: that R0 = T1, R1 = S2; that three rows, the minimum, were computed; and that the stopping criterion was satisfied quickly, since R2 = R1.

Now press |RETURN| to conclude the demonstration. The display (Screen 5) will show the results of all three methods. After viewing the display, press |RETURN| to prepare for the next example.

```
F(X) = 1 + X * X
   A = 0                B = 1

T2  = 1.375           R0 = 1.5
T5  = 1.34            R1 = 1.33333333
T10 = 1.335           R2 = 1.33333333
T20 = 1.3375
T50 = 1.3334           R = 1.33333333

S2  = 1.33333333
S4  = 1.33333333
S10 = 1.33333333
S20 = 1.33333333
S50 = 1.33333333

PRESS RETURN TO CONTINUE
```

Screen 5. The results of all three methods, displayed for comparison.

Example 2. Evaluate

$$\int_1^4 (x^3 + 2x^2 - 3x + 1)dx$$

by all three methods, and compare the results. Use the values for n and ε from Example 1.

Solution. Press |S| on the integration method menu (Screen 2), then press |C| and enter

F(X) = ((X + 2)*X − 3)*X + 1 A = 1 B = 4

Then press |ESC| to exit the change mode. After checking the entries, press |G| for the integration. The value of 86.25 is obtained for each value of n used, with S20 showing a small round-off error. Simpson's rule has returned the exact value of the integral, 86.25, because the integrand is a polynomial of degree less than four.

Press |RETURN|, |T|, and |G| for the trapezoidal approximation. The error in T50 is 86.2671 - 86.25 = .0171, which is less than .02 percent of the integral's true value.

Now find the Romberg approximation. The exact value of the integral is returned quickly, after computation of the minimum number of rows.

Finally, press |RETURN| to compare the results of all three methods. Simpson's rule and Romberg integration are superior to the trapezoidal rule for cubic functions as well as for quadratic functions. The exactness of the Simpson's rule approximation is predictable because the expression for the error (Section 2) has $f^{(iv)}(c)$ as a factor, and the fourth derivative of a cubic function is identically zero.

Now press |RETURN| to return to the method menu for the next example.

Example 3. Evaluate

$$\int_0^4 \sqrt{x}\, dx$$

by all three methods used in this program. Use n = 2, 5, 10, 20, 100 for the trapezoidal rule, n = 2, 4, 10, 20, 100 for Simpson's rule, and ε = .00001 for the Romberg integration.

Solution. Press |T| on the integration method menu. Then press |C| and enter

F(X) = SQR(X)   A = 0   B = 4

Press |RETURN|s to accept the approximations T2, T5, T10, and T20, and then press |1| |0| |0| |RETURN| to set T100 as the last trapezoidal approximation to be found.

After checking your entries, press |G| to start the calculations. The values converge slowly, with T100 = 5.33170358. Since

$$\int_0^4 \sqrt{x}\, dx = (2/3)x^{3/2}\Big|_0^4 = 16/3,$$

the approximation is accurate only to two decimal places.

Press |RETURN|, then |S|, and |C|. Only the last value of n need be changed, so press |RETURN| repeatedly until the cursor reaches this value, and press |1| |0| |0| |RETURN|. Now press |G| to integrate. The convergence is still relatively slow, with S100 = 5.33268385, accurate to three places.

Continue on to obtain the Romberg approximation. This time the full 10 extrapolation rows are computed, with R = 5.333286 accurate to 4 places. Note the substantial amount of computing time. It takes about fifteen seconds to go from R8 to R9. Finally, press |RETURN| to compare the results of all three methods.

Example 4. (A Periodic Function) Evaluate

$$\int_0^{2\pi}(1 + x\cos 20x)dx$$

using the values of n and ε from Example 3.

Solution. Starting from the integration method menu, press |T| and |C|, and then enter

F(X) = 1 + X*COS(20*X)   A = 0   B = 2*PI

If the values of n are correct from Example 3, press |ESC|; otherwise enter new values as necessary. When the entries are checked, press |G| to integrate. Note that poor approximations are returned for n = 2, 5,10, and 20, but that T100 = 6.28318508. The exact value of the integral is 2π, which is 6.28318531 to 8 places.

Now press |RETURN|, |S|, and |G| to find the Simpson's rule approximation, with S100 = 6.28318507 only slightly less accurate than T100.

Try the Romberg routine; a poor result is returned. The reason is that the minimum number of rows to be computed, which has been set at 3 in the program to give fast results in many problems, is too low for this example. It should be noted, however, that Romberg integration with $\varepsilon = 10^{-9}$ yields the final value R = 6.28318533, which is closer to 2π than either T100 or S100.

The gross inaccuracy of the trapezoidal and Simpson approximations for the smaller values of n in this example is explained by the fact that the integrand function oscillates in its amplitude envelope twenty times in the given interval.

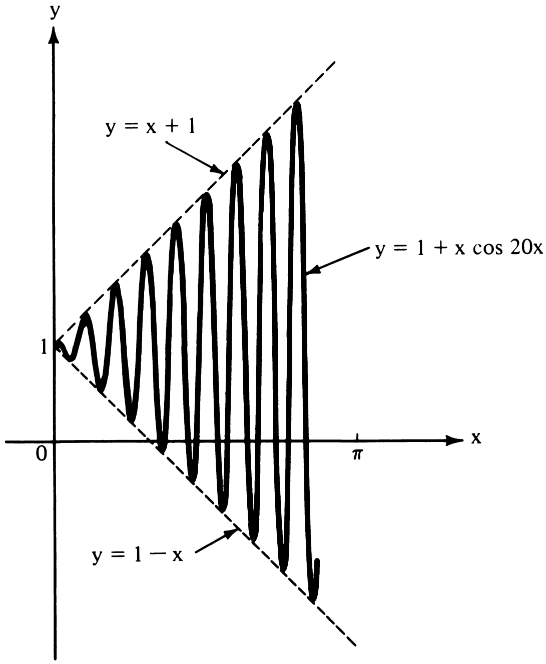


Figure 3. Graph of $y = 1 + x\cos 20x$. If n divides 20, then the sample of function values will be biased because the sampled points between 0 and 2π will all generate relative maxima.

Since the first four values of n all divide 20, the function evaluations give relative maxima. (The results are reasonable for small values of n that do not divide 20.)

Example 5. (A Removable Discontinuity) Evaluate

$$\int_0^1 \sin(x)/x \, dx,$$

using the values of n and ε from Example 3.

Solution. Press |D| on the program menu (Screen 1) and then press |T| for the trapezoidal rule. If you have just come from Example 4, the display will look like the one in Screen 6.

```
INTEGRAND AND LIMITS OF INTEGRATION

  F(X) = SIN(X) / X
     A = 0              B = 6.2831853

TRAPEZOIDAL APPROXIMATIONS TO BE FOUND
T2       T5       T10      T20      T100

REMOVE DISCONTINUITIES
F(A) = 1               F(B) = .841470985

|C|HANGE ENTRY   |G|O ON   |M|ENU   |Q|UIT   |_|
```

Screen 6. The initial trapezoidal display in Example 5.

The only value shown in Screen 6 that needs to be changed for this problem is the value of B. Press |C| and two |RETURN|s to accept F(X) = SIN(X)/X and A = 0, and then enter B = 1. Then accept the T-values with |RETURN|s. Note the new message that appears when the cursor jumps to the value of F(A):

|RETURN| ACCEPT ENTRY
|ESC| AUTOMATIC ENTRY IF F IS CONTINUOUS AT A
ENTRY LIMIT: 12 CHARACTERS

Since

$$\lim_{x \to 0} \frac{\text{SIN(X)}}{\text{X}} = 1 \quad \text{and} \quad \frac{\text{SIN(1)}}{1} = .841470985,$$

the default values are correct. Accept them with |RETURN|s

and press |G| for the computation.

Now run the calculation with Simpson's rule and the Romberg method. Screen 7 shows how the results compare.

```
F(X) = SIN(X) / X
   A = 0                B = 1

F(A) = 1             F(B) = .841470985

T2   = 9.39793285       R0 = .920735492
T5   = 9.45078781       R1 = .946145882
T10  = 9.45832072       R2 = .946083004
T20  = 9.46020325       R3 = .946083071
T100 = 9.4608056
                         R = .946083071
S2   = .946145882
S4   = .946086934
S10  = .946083169
S20  = .946083077
S100 = .94608307

PRESS ANY KEY TO CONTINUE |_|
```

Screen 7. The final screen of Example 5.

Example 6. (Another Removable Discontinuity) Evaluate

$$\int_0^1 2(e^x - 1)/x \, dx$$

using the values of n and ε from Example 5.

Solution. Press |D| from the program menu (Screen 1), then press |T| and |C|, then enter

F(X) = 2*(EXP(X) - 1)/X

If you have just completed Example 5, the values shown for A, B, and n should be correct. Make any necessary changes. This will bring you to F(A). Since

$$\lim_{x \to 0} 2(e^x - 1)/x = 2,$$

enter F(A) = 1 and press |ESC| when the cursor jumps to F(B). The value 3.43656366 should appear on the screen for F(B), since this is 2(e − 1) to 8 places. Carry out the integrations to find T100 = 2.63581264 and S100 = 2.6358043, with the Romberg value returned after only four rows.

Example 7. (An Improper Integral − Substitution) Evaluate

$$\int_{-1}^{1} (\cos x)/\sqrt{1-x^2}\,dx.$$

Solution. Since

$$\lim_{x \to \pm 1} (\cos x)/\sqrt{1-x^2} = +\infty,$$

the discontinuities are not removable. However, the substitutions

$$x = \sin t, \qquad dx = \cos t\,dt$$

yield an equivalent integral that can be evaluated by this program:

$$\int_{-1}^{1} \frac{\cos x}{\sqrt{1-x^2}}dx = \int_{-\pi/2}^{\pi/2} \frac{\cos(\sin t)}{\sqrt{1-\sin^2 t}}(\cos t\,dt) = \int_{-\pi/2}^{\pi/2} \cos(\sin t)dt.$$

Press |C| on the program menu (Screen 1), and choose any of the three integration methods. Enter F(X) = COS(SIN X), A = −PI/2, and B = PI/2, and carry out the integration. The correct value of the integral to 8 places is 2.40393943.

Example 8. (An Improper Integral − Integration by Parts) Evaluate

$$\int_{0}^{1} (\cos x)/\sqrt{x}\,dx.$$

Solution. The problem can be handled using integration by parts. With

$$u = \cos x \qquad\qquad du = -\sin x\,dx$$

$$dv = dx/\sqrt{x} \qquad\qquad v = 2\sqrt{x}.$$

we find

$$\int_0^1 (\cos x)/\sqrt{x}\,dx = 2\sqrt{x}\cos x\Big|_0^1 + 2\int_0^1 \sqrt{x}\sin x\,dx$$

$$= 2(\cos 1 + \int_0^1 \sqrt{x}\sin x\,dx).$$

A Romberg integration with $\varepsilon = 10^{-8}$ yields

$$\int_0^1 \sqrt{x}\sin x\,dx = .3642219$$

to seven places, from which

$$\int_0^1 (\cos x)/\sqrt{x}\,dx = 1.80905$$

to five places.

PROBLEMS

Evaluate the following integrals by all three INTEGRAL EVALUATOR methods, and compare the results. Experiment with larger values of n and smaller values of ε to achieve desired accuracy. Caution: You will have to add parentheses to some formulas as you key them in.

1. $\int_0^2 (1 - 3x)dx$
2. $\int_{-1}^1 (x^3 + 1)dx$
3. $\int_0^2 (x^4 + 1)dx$
4. $\int_0^1 \sqrt{x^2 + 1}\,dx$
5. $\int_{-1}^1 \sqrt{1 - x^2}\,dx$
6. $\int_0^2 \sqrt{x^3 + 1}\,dx$
7. $\int_0^{\pi/2} \cos x\,dx$
8. $\int_0^{\pi} \sin x\,dx$

9. $\int_1^2 \arctan x \, dx$

10. $\int_1^4 \ln x \, dx$

11. $\int_0^1 e^x dx$

12. $\int_0^1 \sec x \, dx$

13. $\int_0^\pi x^4 \sin x \, dx$

14. $\int_0^1 \operatorname{arccot} x \, dx$

15. $\int_0^1 \cosh x \, dx$

16. $\int_1^2 \cot x \, dx$

17. $\int_0^2 [(\cos x - 1)/x] \, dx$

18. $\int_1^2 [(\tan(x - 2))/(x - 2)] \, dx$

19. $\int_0^1 x \ln x \, dx$

20. $\int_1^{10} x^2 \ln x \, dx$

21. $\int_{-1}^0 [(\arctan x)/x] \, dx$

22. $\int_0^1 e^x / \sqrt{x} \, dx$

23. $\int_0^1 x^2 \sqrt{1 - x^2} \, dx$

24. $\int_0^1 1/\sqrt{1 + \cos^2 x} \, dx$

N. Antiderivatives and Direction Fields

1. PURPOSE

This program enables you to study solutions of the differential equation

$$y' = f(x,y)$$

geometrically by constructing their graphs in the direction field in a variety of rectangular regions in the xy-plane.

2. WHAT DIRECTION FIELDS CAN SHOW

The direction field of a first order differential equation

$$y' = f(x,y)$$

often reveals important information about the equation's solutions. It can reveal zones in which the solution curves decrease, or increase, show upper and lower bounds for solution curves, suggest the asymptotic behavior of solutions, and indicate the dependence of solutions on initial conditions. It thus serves as a useful complement to analytic and numerical techniques, and can be an especially valuable source of information when analytic techniques fail.

3. DESCRIPTION

This program plots direction fields of differential equations of the form DY/DX = F(X,Y) inside a rectangular region XMIN < X < XMAX, YMIN < Y < YMAX. You enter the formula for F and choose the bounds for X and Y. Once the field is displayed, you may request the graph through any point in the region. The portion of the solution curve that lies within the region is then shown against the direction field. Several solution curves may be shown at once. At any time, you may clear the screen of solution curves without losing the field elements, or request a new region in which to view the field.

4. STEP BY STEP

Load the program, read the greeting messages, and press |RETURN| to display the equation and domain menu shown in Screen 1.

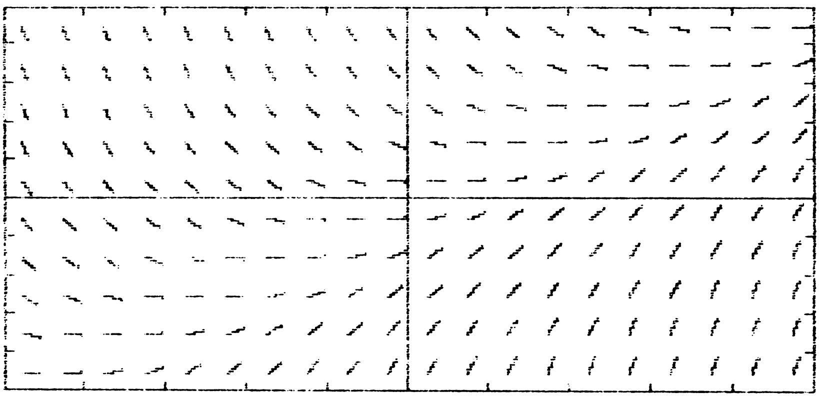
```
EQUATION AND XY-DOMAIN

DY/DX = X - Y
 XMIN = -3              XMAX = 3
 YMIN = -3              YMAX = 3

|C|HANGE ENTRY   |G|O ON   |Q|UIT   |_|
```

Screen 1. The equation and domain menu.

Example 1. The differential equation
DY/DX = X - Y, -3 < X < 3, -3 < Y < 3.

Accept the function and values shown in Screen 1 by pressing |G|, and spend a moment with the resulting display, shown here as Screen 2. This display shows the direction field of DY/DX = X - Y over the chosen region, along with

the visible portions of the X- and Y-axes. Each edge of the screen is divided by marks into ten congruent intervals, each six tenths of a unit long in this case because the rectangle is six X-units wide by six Y-units high.



DY/DX = X - Y

-3 < X < 3 -3 < Y < 3

|S|OLUTION |C|LEAR |N|EW FIELD |Q|UIT

Screen 2. Part of the direction field of DY/DX = X - Y.

Pressing |S| enables you to enter the coordinates of a point through which you wish to plot a solution curve. To see how this works, press |S|, and when the line

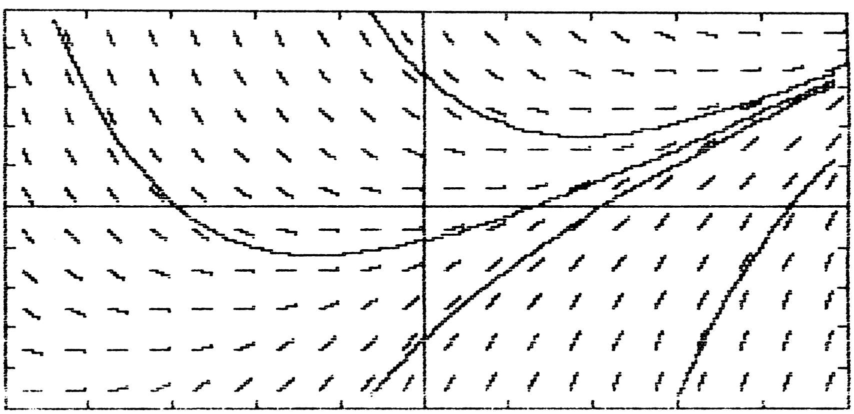
X0 = 0 Y0 = 0

appears near the bottom of the screen press two |RETURN|s. The computer will plot the antiderivative of DY/DX = X - Y that passes through (X0,Y0) = (0,0). After noting the agreement between the solution and the direction field, press |C| to clear the curve from the screen.

Now press

|S| |RETURN| |-| |.| |5| |RETURN|

to plot the solution through the point (0,-.5). Press |S| again and enter the coordinates of (0,-2). Repeat with (0,2) and (2,-2). The display will now look like the one in Screen 3.



DY/DX = X - Y

-3 < X < 3 -3 < Y < 3

X0 = 2 Y0 = -2

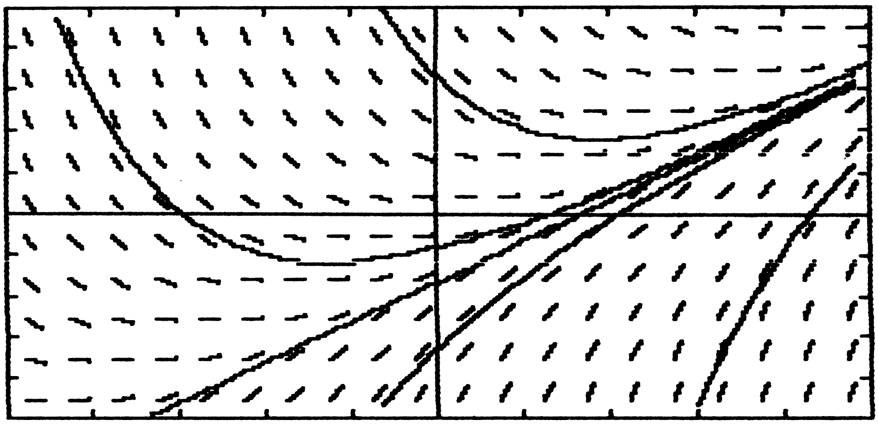
|S|OLUTION |C|LEAR |N|EW FIELD |Q|UIT |_|

Screen 3. Solution curves through the points (0,-.5), (0,-2), (0,2), and (2,-2).

After plotting these solution curves, and perhaps a few others, you may notice that the curves all seem to be asymptotic to a single diagonal line. To explore the possibility of there being a solution of the form Y = MX + B, we substitute Y and its derivative DY/DX = M in the differential equation DY/DX = X - Y. This gives

$$\begin{aligned} DY/DX &= X - Y, \\ M &= X - (MX + B), \\ M &= X - MX - B, \\ (1 - M)X &= M + B. \end{aligned}$$

The last equation here will hold for all X if (and only if) M = 1 and B = −1. The function Y = X − 1 is therefore a solution of the differential equation, and the graph of this function is the line we seek. Add it to the screen by requesting the solution through the point (0,−1).



DY/DX = X − Y
−3 < X < 3   −3 < Y < 3
X0 = 0   Y0 = −1
|S|OLUTION |C|LEAR |N|EW FIELD |Q|UIT |_|

Screen 4. A line of constant slope in a direction field is called an <u>isocline</u>. In this case, the isocline Y = X − 1 is also a solution curve.

As we can see in Screen 4, the solution curve Y = X − 1 divides the direction field into two zones. Solutions through points below this line have slopes greater than 1 and are increasing functions of X. Solution curves through

points above the line Y = X - 1 appear to be concave up, as is confirmed by the inequality

$$y'' = 1 - y' = 1 - (x - y) = y - (x - 1) > 0.$$

For more information about the solutions of this equation, see Article 18.14 of <u>Calculus and Analytic Geometry, Sixth Edition</u>, by G. B. Thomas, Jr. and R. L. Finney (Reading, Mass.: Addison-Wesley Publishing Company, Inc., 1984).

Now press |N| for a new field, and work Example 2.

Example 2. The indefinite integral

$$\int 3x^2 dx$$

is the family of functions $y = F(x) + C$ whose derivatives are $y' = 3x^2$. Plot the direction field for $y' = 3x^2$ in the rectangle $-3 < x < 3$, $-10 < y < 30$, and investigate the solution curves.

Solution. Starting from the equation and domain menu (Screen 1), enter DY/DX = 3*X*X and bounding values for X and Y and watch the direction field develop (Screen 5). The X-marks in Screen 5 are still .6 apart, but the Y-marks now represent 40/10 = 4 units. Press

|S| |0| |0| |RETURN|

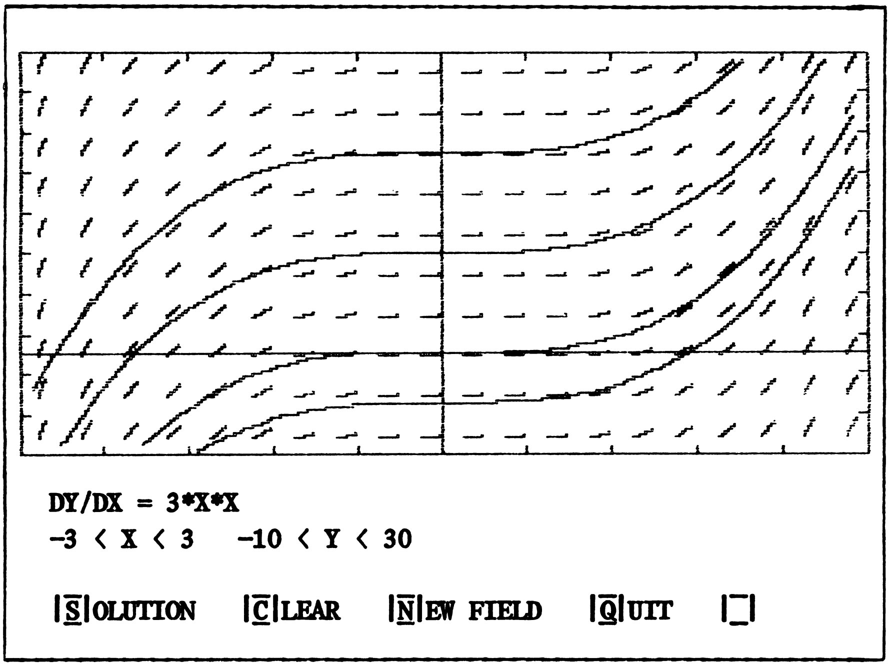
to see the solution through the origin. Then repeat the process to add the curves through the points (0,-5), (0,10) and (0,20). These curves all belong to the family

$$\int 3x^2 dx = x^3 + C,$$

and correspond to taking C = 0, -5, 10, and 20.

4. OTHER FEATURES

To stop a plot in progress, press |ESC|. You may then quit (|Q|) or return to the function screen (|N|), as desired.



Screen 5. The field for DY/DX = 3*X*X, with solution curves through (0,0), (0,-5), (0,10), and (0,20).

The program will not draw solution curves through off-screen points. It will also ignore random keystrokes when the direction field is first displayed, responding only to |S|, |C|, |N|, and |Q|.

The program accepts "PI" for π in function formulas (as in COS(PI*X)), window parameters, and point coordinates.

The left and right arrow keys |<-| and |->|, can be used to edit inputs.

PROBLEMS

In Problems 1–15 plot the direction field of the differential equation in the given window. Then plot the solution curves that pass through the given points.

1. a) $y' = x + y$, $-3 < x < 3$, $-3 < y < 3$
   Points: $(-3,0)$, $(\pm 2,0)$, $(\pm 1,0)$, $(0,0)$
   b) What isocline is also a solution curve?

2. a) $y' = x$, $-3 < x < 3$, $-3 < y < 3$
   Points: $(-3,0)$, $(\pm 2,0)$, $(\pm 1,0)$, $(0,0)$
   b) Identify the isoclines (lines along which $y' = \text{const.}$).

3. a) $y' = y$, $-3 < x < 3$, $-3 < y < 3$
   Points: $(0,.1)$, $(0,.2)$, $(0,1)$, $(0,-.5)$
   b) Identify the isoclines (lines along which $y' = \text{const.}$).

4. $y' = \frac{1}{x}$, $-1 < x < -.1$, $-1 < y < 1$
   Points: $(-.5,0)$, $(-.2,0)$, $(-.3,0)$

5. $y' = \frac{y}{x}$, $-3 < x < 3$, $-3 < y < 3$
   Points: $(1,1)($ $(1,-1)$, $(1,2)$
   Identify the solution curves.

6. $y' = xy$, $-3 < x < 3$, $-3 < y < 3$
   Points: $(0,0)$, $(0,\pm 1)($ $(0,\pm 2)$

7. $y' = 2x - y$, $-3 < x < 3$, $-3 < y < 3$
   Points: $(0,\pm 2)$, $(0,\pm 1)$, $(0,0)$

8. $y' = x^2 + y^2$, $-3 < x < 3$, $-3 < y < 3$
   Points: $(0,\pm 1)$, $(0,1.5)$, $(0,2)$, $(0,3)$, $(2,0)$

9. $y' = (x + y)^2$, $-3 < x < 3$, $-3 < y < 3$
   Points: $(0,0)$, $(-1,1)$, $(1,0)$

10. $y' = y^2 + 1$, $-6 < x < 6$, $-6 < y < 6$
    Points: $(0,0)$, $(0,\pm 3.14)$

11. $y' = 4y(1 - y)$, $0 < x < 2$, $-0.5 < y < 2$
    Points: $(0,.1)$, $(0,.01)$, $(0,-.1)$

12. a) $y' = (y - 2)/(x - 1)$, $1.1 < x < 3$, $-3 < y < 4$
    Points: (2.0), (2.5,0), (2.5,2), (1.2,0), (1.5,0)
    b) Repeat (a) for $-3 < x < .9$, $-3 < y < 4$.
    c) The isoclines of this direction field are the lines $(y - 2)/(x - 1) = m$. Through what point do they all pass?

13. $y' = (x - y)/x$, $.1 < x < 3$, $-3 < y < 3$
    Points: (1,0), (.5,0), (1,1), (2,1)

14. $y' = (4x + 3y)/(3x + y)($ $-3 < x < 3$, $-3 < y < 3$
    Points: $(\pm 1,0)$, $(\pm 2,0)$

15. $y' = (\cos \pi x)/\sqrt{y}$, $-1 < x < 2$, $.1 < y < 2$
    Points: (.5,1), (.5,1.5), (.5,.5)

16. Numerical antiderivative graphers sometimes encounter difficulty when the slope y' at a point on an integral curve is large. Here is an example. A straight forward integration shows that the solutions of the equation $yy' = 1$ are given by the formula $y^2 = 2x + C$. The solution curves are parabolas symmetric about the x-axis. When the computer program described in this chapter is used to plot solutions of the nearly equivalent differential equation $y' = 1/y$ in a region about the x-axis, however, the curves go somewhat astray. To find out what happens, plot the direction field for DY/DX = 1/Y over the region $-3 < X < 3$, $-3 < Y < 3$ and request the solution curves through the points (1,1) and (1,-1).

Each indefinite integral $\int f(x)dx$ in Problems 17-21 is a family of functions $y = F(x) + C$. Plot the direction field for $y' = f(x)$ in the given rectangle. Then plot the members of the family $y = F(X) + C$ that pass through the given points.

17. $\int \cos 2x\, dx$, $0 < x < 2\pi$, $-4 < y < 4$
    Points: (0,0), (2,2), (2,3), (0,-3), (0,-.6)

18. $\int \sqrt{2x + 1}\,dx$, $.5 < x < 12$, $0 < y < 25$
    Points: $(.5,1)$, $(.5,2)$, $(.5,10)$

19. $\int (3 \sin 2x + 4 \cos 3x)dx$, $0 < x < 2\pi$, $-5 < y < 5$
    Points: $(0,0)$, $(0,-4)$

20. $\int (x - 7)^3 dx$, $5 < x < 9$, $-3 < y < 3$
    Points: $(7,0)$, $(7,-2)$, $(6,-2)$, $(5.5,-2.5)$

21. a) $\int (x - 3 \sin 4x)dx$, $-5 < x < 5$, $-9 < y < 9$
    Points: $(0,1)$, $(-5,0)$
    b) Repeat (a) for $-7 < x < 7$, $-20 < y < 20$
    c) Repeat (a) for $-10 < x < 10$, $-50 < y < 50$

O. Partial Fraction Integration Problems

1. PURPOSE

This program enables you to practice integrating rational functions by the method of partial fractions.

2. DESCRIPTION

The program generates partial fraction integration problems one at a time for you to solve. You may determine in advance whether the problems are to contain linear factors only or quadratic factors as well. You may also decide whether to allow repeated roots.

Once these choices have been made, the screen will display the integral of an appropriate rational function of X along with a box in which your answer will appear as you key in the value of the integral. If you wish to change what you have keyed in, you may edit character by character or press |ESC| to clear the box and start the answer over. When you are satisfied with your answer, press |RETURN| to signal that your answer is complete. The computer will then say whether your answer is right or wrong.

If your answer is right you may press |A| for another problem of the same kind, press |M| to call up the problem type menu to change the problem type, or press |Q| to quit.

If your initial answer is wrong, the correct partial fraction decomposition of the integrand will appear on the

screen. You may then enter a new answer for the value of the integral. If your second answer is wrong, the value of the integral will appear on the screen and you may proceed to press |A|, |M|, or |Q| as before.

If you need help, type HELP in the current answer box. The computer will treat this as a wrong answer and make one of the two responses just described.

The computer keeps a running score of the number right out of the number tried.

You will need a pencil and paper.

3. STEP BY STEP

Load the program from the disk menu, read the greeting message, and press |RETURN| to see the problem type menu:

```
                 <PROBLEM TYPES>

          1... LINEAR FACTORS ONLY
          2... QUADRATIC FACTORS ONLY
          3... LINEAR AND QUADRATIC FACTORS

          4... QUIT, LEAVE PROGRAM

             PRESS 1, 2, 3 OR 4
```

Screen 1. The problem type menu lets you choose the kind of problem you want to practice with.

To begin cautiously, you might press |1| for linear factors only. Once your selection is made, the prompt

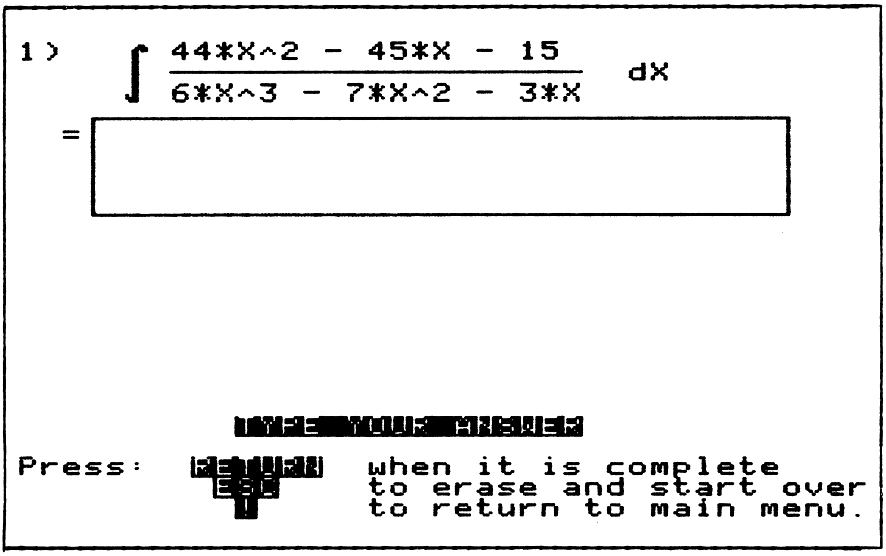
DO YOU WANT TO ALLOW
REPEATED FACTORS? (Y/N)

will appear on the screen. When it does, press |Y| or |N| to indicate your choice of yes or no.

When your first problem appears, work out the partial fraction expansion of the integrand on a piece of paper and evaluate the integral. Then check your answer by typing it into the computer and pressing |RETURN|. The answer will appear on the screen as you type. You can correct typing mistakes by backing up and striking over (press |<-| to back up and |->| to move forward again). You can also press |ESC| to clear the entire box and start over. Only when you are satisfied with how the answer reads on the screen should you press |RETURN|.

Your problem sequence will probably differ from the one we encountered while preparing the present chapter because the order in which the problems are presented is random.

Screen 2 shows our first problem. When we asked for help, the computer responded by expanding the integrand by partial fractions (Screen 3). We then worked out the integral on paper, keyed in our answer, and were told it was correct (Screen 4).



Screen 2. The computer is waiting for an answer.

```
1 )      ⌠ 44*X^2 - 45*X - 15
         ⌡ ------------------------   dX
           6*X^3 - 7*X^2 - 3*X
  = HELP

         ⌠     2           4
  =      ⌡ ---------  + ---------  + ( 5/X )   dX
           2*X  - 3     3*X  + 1
  =

Press:        when it is complete
              to erase and start over
              to return to main menu.
```

Screen 3. The hint after the first wrong answer.

```
1 )      ⌠ 44*X^2 - 45*X - 15
         ⌡ ------------------------   dX
           6*X^3 - 7*X^2 - 3*X
  = HELP

         ⌠     2           4
  =      ⌡ ---------  + ---------  + ( 5/X )   dX
           2*X  - 3     3*X  + 1
  = LOG( ABS( 2*X  - 3 ))
    + ( 4/3 )*LOG( ABS( 3*X  +1 ))
    + 5*LOG( ABS( X ))

 SCORE:   .5 right of 1 tried = 50%
PRESS:   Another problem, Menu, Stop
```

Screen 4. Answering the first problem correctly on the second try gives a score of 50%.

What about the arbitrary constant? The computer will accept any constant or letter you wish to add to the antiderivative of the integrand, but does not require you to add one. In our next answer we added an arbitrary constant C:

3) $$\int \frac{6}{2*X + 1} \, dX$$

= `3*LOG(ABS( 2*X+1 )) + C` RIGHT

Screen 5. The program accepts arbitrary constants.

The answer you type in may be "stacked," as in Screen 4, or "strung out" as in the next display.

4) $$\int \frac{-7*X\text{^}2 - 6*X + 18}{3*X\text{^}3 + 15*X\text{^}2 + 18*X} \, dX$$

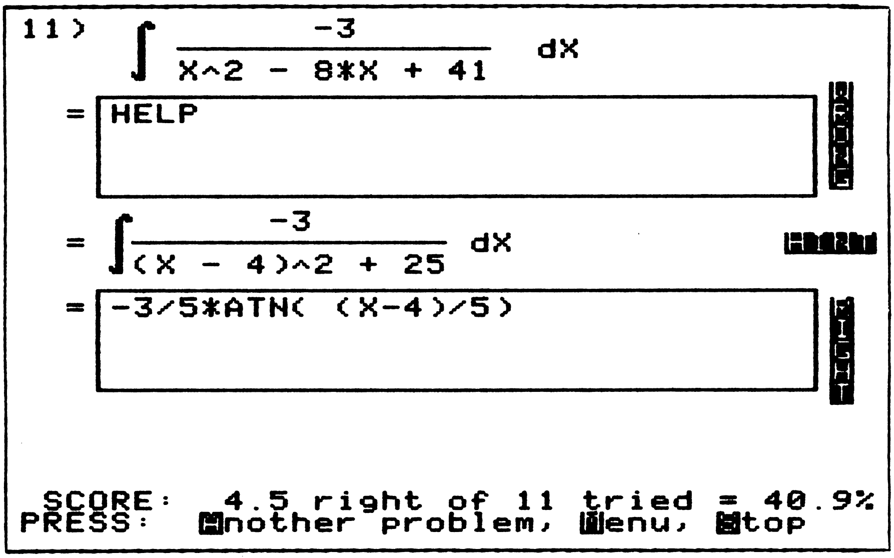
= `HELP`

$$= \int \frac{-3}{X + 3} + \frac{-1}{3*X + 6} + (1/X) \, dX$$

= `-3*LOG(ABS(X+3))+(-1/3)*LOG(ABS(3*X+6))+LOG(ABS(X))` RIGHT

Screen 6. You do not have to worry about line breaks when you type your answer.

When we asked for help with the next problem, the computer responded by showing how to write this particular denominator as the sum of two squares, so we knew we had an arc tangent to deal with.



Screen 7. An arc tangent.

4. CONCLUDING REMARKS

In Applesoft BASIC, the natural logarithm of X is LOG(X), the absolute value of X is ABS(X), and the arc tangent of X is ATN(X). See Appendix 2 for other function formulas.

The program separates the two steps in partial fraction integration problems, the decomposition and the integration. The two-step/two-try format enables you to practice the entire process or work on either part separately.

P. Conic Sections

1. PURPOSE

This program enables you to see the effects of selected rotations and translations of the coordinate axes on the equations of lines, circles, parabolas, ellipses, and hyperbolas in the cartesian plane.

2. DESCRIPTION

When a conic (conic section) is selected from the program menu, the computer displays the axes shown in Fig. 1

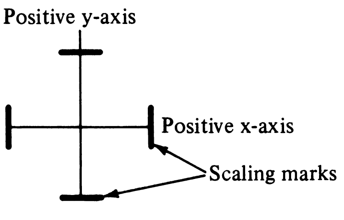


Figure 1. The coordinate axes on the display screen are movable.

in a standard position with respect to the selected conic. We call this position the <u>initial</u> <u>position</u> of the axes. The equation of the conic in the cartesian coordinate system *defined* by the axes appears near the bottom of the screen.

The axes may be rotated clockwise or counterclockwise in increments of fifteen degrees, and may be translated right, left, up, or down in steps that are the size of the current axis scaling unit. (We shall have more to say about this unit later.)

The conic remains stationary as the axes move. After each motion, the screen displays the equation of the conic in the new coordinate system.

3. STEP BY STEP

Load the program from the disk menu, read the greeting message, and go on to the menu shown in Screen 1.

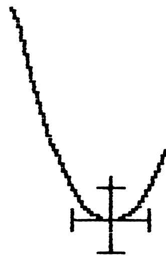
```
|L|INE
|C|IRCLE
|P|ARABOLA
|E|LLIPSE
|H|YPERBOLA
|Q|UIT
```

Screen 1. The conic section menu.

Press |P| to request the parabola shown in Screen 2. Its equation in the initial coordinate system is Y + 0 = (X + 0)^2, or $Y = X^2$.

Scaling. The statement UNIT = 1 in Screen 2 tells us that the marks on the axes currently represent a distance of one unit from the origin. To investigate the effect of changing scale, press |S|. The writing at the bottom of the screen will immediately change to

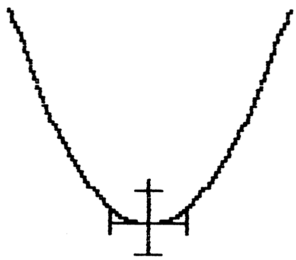
NEW UNIT = |1|
ENTER A NUMBER BETWEEN .1 AND 25.



Y + 0 = (X + 0)∧2

|U|P
(|P|OS, |N|EG) ROTATION |L|EFT |R|IGHT
|M|ENU |S|CALE |A|XIS UNIT = 1 |_| |D|OWN

Screen 2. The equation of the parabola in the initial coordinate system is Y + 0 = (X + 0)∧2.



Y + 0 = (X + 0) ∧ 2

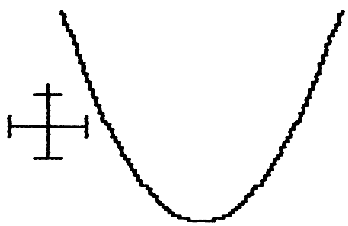
|U|P
(|P|OS, |N|EG) ROTATION |L|EFT |R|IGHT
|M|ENU |S|CALE |A|XIS UNIT = .5 |_| |D|OWN

Screen 3. Changing the scale does not change the parabola's equation. It changes the "magnification" of our view of the parabola.

Enter .5 by pressing |.| |5| |RETURN|. The equation of the parabola will reappear as the parabola is redrawn in the new scale (Screen 3).

The marks on the axes now represent a half-unit, instead of a unit, and the parabola appears larger than it did before. The equation of the parabola has not changed, however. We have not moved the axes, but only changed the scale to take a closer look at the region around the origin.

Translation. Now press |U| three times (with pauses in between) and |L| four times to translate the axes three units up and four units to the left. Watch what this does to the equation of the parabola (Screen 4).



Y + 1.5 = (X - 2)^2

|U|P

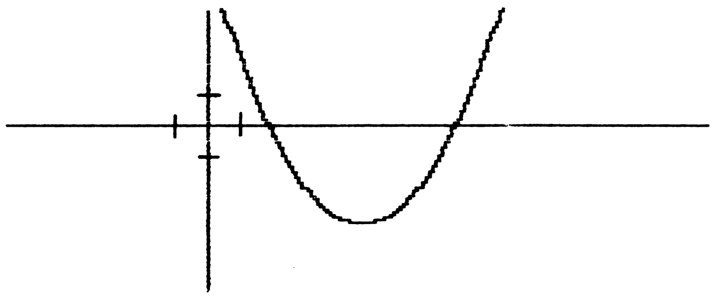
(|P|OS, |N|EG) ROTATION |L|EFT |R|IGHT

|M|ENU |S|CALE |A|XIS UNIT = .5 |_| |D|OWN

Screen 4. The result of translating the axes to a position three units above and four units to the left of their initial position.

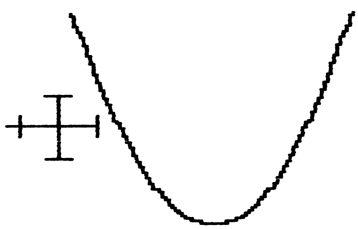
The equation of the parabola shown in Screen 4 is <u>not</u> Y + 3 = (X - 4)^2, but rather Y + 1.5 = (X - 2)^2. The vertical component of the axis translation was (+3)(.5) = 1.5, three times the current scale unit of .5. The horizontal component of the axis translation was (-4)(.5) = -2, accomplished by taking four steps of size .5 to the left.

Now press |A| to extend the axes across the screen. This will show their geometric relation to the parabola more clearly (Screen 5).



```
Y + 1.5 = (X - 2)^2
                                             |U|P
(|P|OS, |N|EG) ROTATION                 |L|EFT  |R|IGHT
 |M|ENU |S|CALE |A|XIS UNIT = .5      |_|  |D|OWN
```

Screen 5. The axes extended.



```
X + 1.5 = (Y + 2)^2
                                             |U|P
(|P|OS, |N|EG) ROTATION                 |L|EFT  |R|IGHT
 |M|ENU |S|CALE |A|XIS UNIT = .5      |_|  |D|OWN
```

Screen 6. The result of rotating the axes in Screen 5 a half-turn counterclockwise, or +90°.

Rotation. To continue the demonstration, press |P| six times, pausing after each press to read the parabola's equation in the new reference frame. Each keypress rotates the axes $+15^{\circ}$, or fifteen degrees counterclockwise. The net result is a half-turn counterclockwise, as shown in Screen 6.

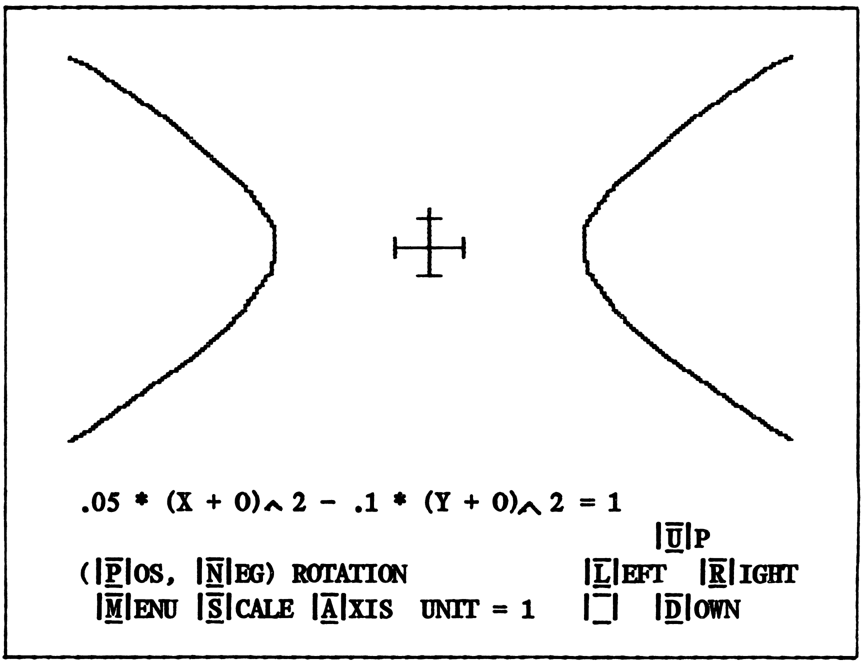
Press |M| to return to the conic section menu, and select the program's standard hyperbola (Screen 8) by pressing |H|. The hyperbola's equation,

$$\frac{x^2}{20} - \frac{y^2}{10} = 1,$$

is displayed as

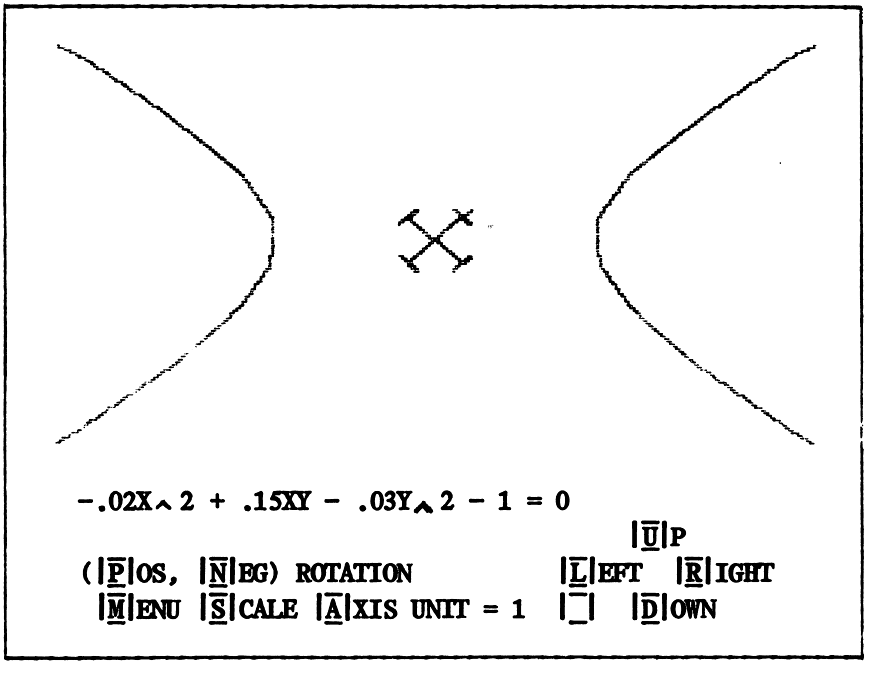
.05 * (X + 0)^2 - .1 * (Y + 0)^2 = 1

to fit it on a single line. The scale has automatically returned to UNIT = 1.



Screen 7. The initial coordinate system for the hyperbola.

To conclude the demonstration, press |N| three times, pausing after each keypress to read the hyperbola's new equation. The net result of the three negative fifteen-degree rotations will be a -45° rotation, or a rotation of forty-five degrees clockwise, as indicated in Screen 8.



Screen 8. The equation of the hyperbola after the initial coordinate system has been rotated -45°.

For a final view of the hyperbola, press |A| to extend the axes to the limits of the display.

PROBLEMS

In Problems 1–5, display the conic by pressing the key indicated on the conic menu of the program. Then display the conic again with the given scale units.

1. Line: UNIT = 2, .5, .1
2. Circle: UNIT = 2, .5
3. Parabola: UNIT = 2, .5, .1
4. Ellipse: UNIT = 2, .5
5. Hyperbola: UNIT = 4, 2, .75

In Problems 6–10, experiment to find a sequence of axis rotations and/or translations that will move the coordinate axes from their initial position to a position in which the given conic has the given equation. In each case, use the default axis scale, UNIT = 1.

6. The line $y = 0$:

   a) $y = x - 1$ b) $y = -x - 1.41$
   c) $x = -1$ d) $y = x + 1.41$

7. The circle $x^2 + y^2 = 9$:

   a) $(x - 2)^2 + y^2 = 9$ b) $(x - 2)^2 + (y - 2)^2 = 9$
   c) $(x - 2.83)^2 + y^2 = 9$
   d) $(x - 2)^2 + (y + 2)^2 = 9$

8. The parabola $y = x^2$:

   a) $y + 3 = x^2$ b) $-y + 3 = x^2$
   c) $y - 3 = x^2$
   d) $.5x^2 - xy + .5y^2 - .71x - .71y = 0$

9. The ellipse $.05x^2 + .1y^2 = 1$:

a) $.1x^2 + .05y^2 = 1$
b) $.05x^2 + .02xy + .1y^2 - 1 = 0$
c) $.05(x + 1)^2 + .1(y + 1)^2 = 1$
d) $.05(x + 2)^2 + .1(y - 3)^2 = 1$

10. The hyperbola: $.05x^2 - .1y^2 = 1$:

a) $-.1x^2 + .05y^2 = 1$
b) $-.1x^2 + .05(y - 1)^2 = 1$
c) $-.1(x + 4)^2 + .05(y + 10)^2 = 1$
d) $.05(x - 1)^2 - .1y^2 = 1$

11. Starting with the axes in their initial position, say what effect a 180^o rotation has on the initial equation of the (a) line, (b) circle, (c) parabola, (d) ellipse, (e) hyperbola.

12. Starting with the axes in their initial position, say what effect a 90^o rotation has on the initial equation of the (a) line, (b) circle, (c) parabola, (d) ellipse, (e) hyperbola.

In Problems 13-17, identify the conic and move the coordinate axes on the screen to a position in which the conic has the given equation. Problem 17 requires a change of scale.

13. $.05(x - 7)^2 - .1(y)^2 = 1$
14. $y = x - 7.07$
15. $.1(x + 3)^2 + .05y^2 = 1$
16. $y = (x - 11)^2$
17. $y + .1 = x^2$

Q. Sequences and Series

1. PURPOSE

This program enables you to look for numerical and graphical indications of the convergence or divergence of an infinite sequence or series. It will also plot the initial terms of a series and the series' partial sums in a common graph.

2. DESCRIPTION

The program generates terms of one or two sequences and graphs the values of successive terms while you watch. Pressing the space bar stops or restarts the plot. A numerical display shows the values of the terms currently being plotted. The initial number of terms is fifty. When the graph is completed, you may change scale, return to the menu, or request the next fifty terms.

Sequences are entered either by giving recursive formulas and first terms or by giving formulas for Nth terms and specifying initial values of N.

3. STEP BY STEP

Load the program from the disk menu, read the greeting message, and press |RETURN| to begin. The computer will ask whether you want one sequence, or two (Screen 1).

|<INPUTTING SEQUENCES>|

How many sequences? (1 or 2) _

Screen 1. The choice here determines the number of sequences the computer will ask you to define, and will subsequently display.

Example 1. Graphing the Alternating Harmonic Sequence.

The prompt in Screen 1 asks you to press |1| or |2|. Press |1|, and then |RETURN| to accept the default formula,

A(N) = ((-1)^(N + 1))/N,

for the Nth term of the alternating harmonic sequence. After a brief pause, the question

Start sequence at N = ? _

will appear toward the middle of the screen. To start the alternating harmonic sequence at its first term, 1, press |1| |RETURN|. After flashing the message

<ONE MOMENT PLEASE>,

the lines

VERTICAL SCALE
YMIN = -1
YMAX = 3/2

will be added to the bottom of the screen. This is the point at which you determine the vertical dimensions of the horizontal strip in which the points (N,A(N)) will be plotted. The display should now look like the one in Screen 2.

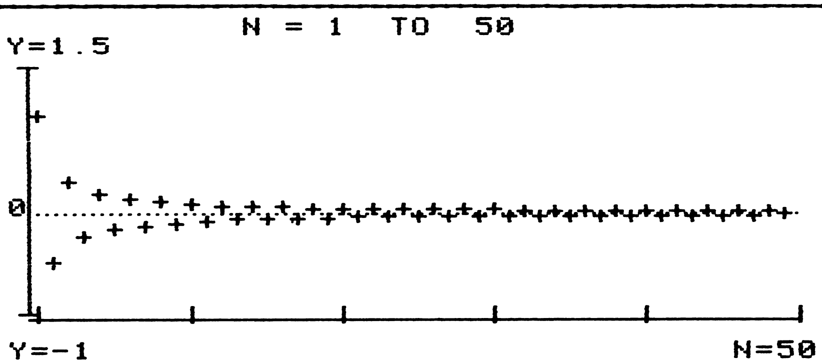
|<INPUTTING SEQUENCES>|

How many sequences? (1 or 2) 1
DEFINE A(N) = ((-1)^(N + 1))/N

Start sequence at N = ?1

VERTICAL SCALE
YMIN = -1
YMAX = 3/2

Screen 2. With a formula for A(N) entered, along with the initial value of N, it remains only to set the vertical scale for the graph.



(+) A(N) = ((-1)^(N+1))/N
A(50) = -.02
Press [illegible] for 50 more values
Change scale or Menu

Screen 3. The points (N,A(N)) are plotted with plus marks (+).

Press |RETURN| twice to accept the present values. The computer will draw axes and plot the points (N,A(N)) for

N = 1 to 50. You will see the points plotted from left to right across the screen as a numerical counter on the lower left runs through the values of A(1), A(2), . . ., A(50). Press any key to interrupt the plotting or start it again. When all fifty terms have been plotted, the message PRESS ANY KEY TO STOP/RESTART PLOT, which was present at the bottom of the screen as the graph evolved, will be replaced by a short menu. The final display will look like the one in Screen 3.

Once the display in Screen 3 has appeared, pressing |C| will enable you to change the values of YMIN and YMAX and replot. Pressing |RETURN| will plot the terms from N = 51 to 100. Pressing |M| will call up the options menu shown in Screen 4. To conclude the example, press |M|.

|<CURRENT DISPLAY>|

A(N) = ((-1)^(N + 1))/N

N = 1 TO 50
YMIN = -1
YMAX = 1.5

|<OPTIONS>|

|1| SEE CURRENT DISPLAY
|2| GRAPH 50 MORE VALUES
|3| CHANGE SCALE
|4| CHANGE SEQUENCES
|5| QUIT

Press the number of your option choice.

Screen 4. The options menu.

Press |4| to change sequences, and work through the next example.

Example 2. <u>Graphing the Partial Sums of the Alternating Harmonic Series.</u>

Starting from the operations menu displayed in Screen 4, press |4| to change sequences. Then press |2| when asked "How many sequences?" and press |RETURN| to accept the formula

A(N) = ((-1)∧(N + 1))/N

for the Nth term of the alternating harmonic sequence. The line

B(N) = B(N - 1) + A(N)

will be added to the screen. It shows the recursion formula for generating the sequence B(N) of partial sums of the sequence A(N), or, as you may wish to call it, the sequence of partial sums of the series $\sum$ A(N).

Press |RETURN| to accept the formula for B(N). When the prompt

Start sequence at N = ?_

appears, press |1| |RETURN| to indicate that the summation should begin with N = 1. The computer will then ask for the value of B(0) by adding

INITIAL VALUES
B(0) = 0

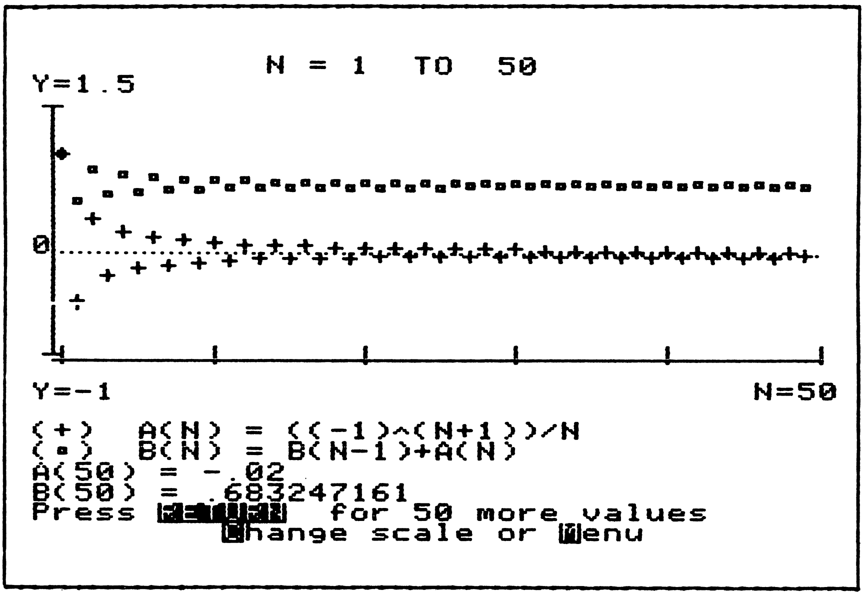
to the screen. Why B(0)? The formula for B(N) defines B(N) in terms of B(N - 1). In particular, B(1) is defined in terms of B(0) as

B(1) = B(0) + A(1).

You must enter a value for B(0) before the computer can generate B(1) and the subsequent values of B(N).

Press |RETURN| once, to accept B(0) = 0, and twice more to accept the current values of YMIN = -1 and YMAX = 3/2. The computer will then plot the alternating harmonic sequence A(N) and the sequence of its partial sums B(N) in a common graph for N = 1 to 50. Screen 5 shows the completed

display.



Screen 5. The alternating harmonic sequence A(N) and the partial sums B(N) of the series $\sum$ A(N) plotted for N = 1 to 50 in a common graph.

The points (N,A(N)) are shown in Screen 5 with plus marks (+); the points (N,B(N)) are shown with small squares (□). The display strongly suggests that the partial sums B(N) approach a limit and lie alternately above and below this limit as N increases. Indeed, averaging B(50) and B(49) = B(50) − A(50) gives

$$\begin{aligned} B(49) + B(50) &= B(50) - \frac{A(50)}{2} \\ &= .683247161 - \frac{-.02}{2} \\ &= .693247161, \end{aligned}$$

which agrees with ln 2 = .6931471681, the sum of the alternating harmonic series, to three decimal places.

Now press |M| to prepare for the next example.

4. RECURSIVE DEFINITIONS

The sequence B(N) = B(N - 1) + A(N) of partial sums of the alternating harmonic series in Example 2 was defined by a formula that calculated B(N) in part from the value of the term B(N - 1). Definitions that define the Nth term of a sequence from a formula that involves one or more preceding terms are called <u>recursive</u> definitions.

The program can accept formulas that define A(N) in terms of A(N - 1) to A(N - 4). Definitions of B(N) can use A(N), A(N - 1), ... , A(N - 4) as well as B(N - 1), ... , B(N - 4).

Note, however, that definitions of B(N) cannot use A(N + 1), so that the ratio B(N) = ABS(A(N + 1)/A(N)) gives nothing. Instead, we use B(N) = ABS(A(N)/A(N - 1)).

In the next example we define the Fibonacci sequence by the formula

A(N) = A(N - 1) + A(N - 2), A(1) = 1, A(2) = 1,

starting with N = 3.

Example 3. <u>The Fibonacci sequence</u>.

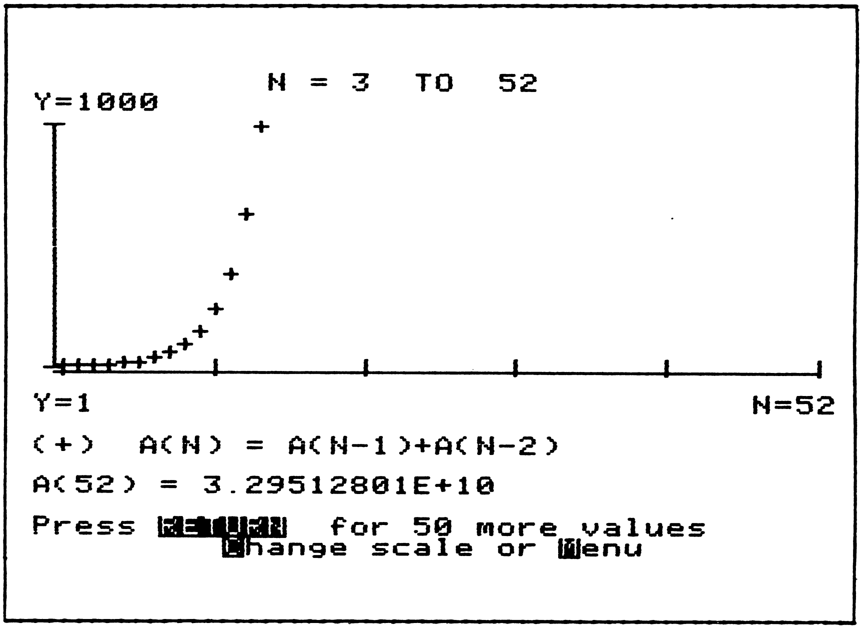
Press |4| on the options menu and press |1| when asked "How many sequences?" Then type in the defining formula

A(N - 1) + A(N - 2)

and press |RETURN|. After the formula has been accepted, press |3| |RETURN| to start the sequence with N = 3. Then enter

A(1) = 1, A(2) = 1, YMIN = 1, YMAX = 1000

by keying in the numbers one at a time followed by |RETURN|s. The plot will begin with N = 3, and the points will rise rapidly after N = 9 and "go off the screen" at N = 17. The numerical calculations will continue, however, through N = 52.



Screen 6. The Fibonacci sequence grows so rapidly that the terms soon exceed YMAX = 1000.

To conclude the example, press |C| to change scale. The display will then change to

|<CURRENT DISPLAY>|

A(N) = A(N - 1) + A(N - 2)

N = 3 TO 52
YMIN = 1
YMAX = 1000

|<CHANGE SCALE>|

YMIN = 1
YMAX = 1000

Screen 7. Changing the vertical scale.

Press |RETURN| to accept YMIN. Then type 3.3E+10 and press |RETURN| to enter a value of YMAX that exceeds A(52). The lines

```
Use new YMIN and YMAX values to —
|R|  REPLOT STARTING AT N = 3
|P|  PLOT STARTING AT N = 53
|M|  RETURN TO THE OPTIONS MENU
        Press |R| or |P| or |M|    |_|
```

will appear at the bottom of the screen. Press |R| and watch the graph develop.

PROBLEMS

Graph the first fifty terms of the sequences in Problems 1-11. Start with N = 1 unless another starting value is given, and use the indicated scale for Y.

1. A(N) = (-1)^(N + 1), $-2 \leq Y \leq 1$
2. A(N) = 1 - (1/N), $0 \leq Y \leq 1$
3. A(N) = ((-1)^(N + 1))*(1 - (1/N)), $-2 \leq Y \leq 1$
4. A(N) = 1 + ((-1)^N)/N, $-7 \leq Y \leq 7$
5. A(N) = (2*N + 1)/(1 - 3*N), $-3 \leq Y \leq 2$
6. A(N) = N*SIN(50/N), $-20 \leq Y \leq 50$
7. A(N) = (LOG(N/2))/(N/2), $0 \leq Y \leq .5$
8. A(N) = N^(1/N), $1 \leq Y \leq 1.5$
9. A(N) = (N*N + 20*N + 1)/(2*N*N + 5), $0 \leq Y \leq 4$
10. A(N) = SQR(4*N + 20)/SQR(N - 1), $1 \leq Y \leq 5$. Start with N = 2.
11. A(N) = (1 + (1/N))^N, $2 \leq Y \leq 3$. When the first fifty terms have been plotted, press |RETURN| to see the next fifty. The convergence is quite slow. To ten digits, e = 2.718281828.

12. To see a dramatic display of the results of round-off and truncation error on a computation, investigate what happens when you try to speed the convergence of the

sequence in Problem 11 by calculating every 2^Nth term. To do this, enter A(N) = (1 + (1/2∧N))∧(2∧N), start with N = 1, and take $2 \leq Y \leq 3$. The convergence toward e will look good at first, but the values calculated for A(N) will behave erratically soon after N = 20.

Problems 13–18 use recursive definitions.

13. **Growth rates.** A function f(N) grows slower than a function g(N) as N–>∞ if $\lim_{N \to \infty} f(N)/g(N) = 0$. If f grows slower than g, we also say that g grows faster than f. The function g(N) = N! grows faster than the function $f(N) = 2^N$. To see early evidence of this fact, define the quotient $2^N/N!$ recursively by setting A(N) = A(N − 1)*2/N with A(0) = 1, and display fifty terms starting at N = 1. Scale: $0 \leq Y \leq 2$.

14. **Heron's method for approximating square roots.** Heron, an Alexandrian mathematician who lived sometime between 100 BC and 100 AD, approximated the square root of a positive number C by calculating successive terms of the sequence A(N) = .5(A(N − 1) + C/A(N − 1). This is the same sequence that is generated by applying the Newton–Raphson method to the function $F(X) = X^2 - C$. Start with N = 1 and A(0) = 1, and use the value given for A(50) to estimate
a) $\sqrt{2}$ b) $\sqrt{3}$ c) $\sqrt{9}$ d) $\sqrt{PI}$.
To ten digits, $\sqrt{2} = 1.414213562$, $\sqrt{3} = 1.732050808$, $\sqrt{PI} = 1.772453851$.

15. **Rapid approximation of π/2.** The sequence A(N) = A(N − 1) + COS(A(N − 1)) with A(0) = 1 approximates $\pi/2 \doteq 1.57079633$ after just a few terms. Try it. To what value do the terms A(N) appear to converge if
a) A(0) = 4? b) A(0) = 5? c) A(0) = −1?

16. **Fixed points of functions.** Under circumstances described in Chapter K, Picard's Fixed Point Method, the sequence A(N) = F(A(N − 1)) converges to a solution of the equation F(X) = X. The convergence depends,

among other things, upon choosing an appropriate value for A(0), and, of course, upon whether a solution exists. The equation COS(X) = X does have a solution. Approximate it by setting A(N) = COS(A(N - 1)) with N = 1, A(0) = 0, 0 $\leq$ Y $\leq$ 1.

17. **Geometric series.** Setting A(N) = A(N - 1)*R, B(N) = B(N - 1) + A(N), A(0) = 1, and B(0) = 1 and starting with N = 1 will calculate successive partial sums B(N) of the geometric series $1 + R + R^2 + \ldots$, and thereby generate a sequence converging to the number 1/(1 - R). Find the value given by the program for B(50) for the following values of R:
a) 1/2 b) 1/3 c) 4/5 d) -1/10.

18. The series $\sum_{N=1}^{\infty}$ 1/(N*(N - 1)) converges to 1. See page 613 of Thomas and Finney's Calculus and Analytic Geometry, Sixth Edition, (Reading, Mass.: Addison-Wesley Publishing Company, Inc., 1984). The progress of the partial sums toward 1 is rapid at first, but slows down noticeably as N increases. Take A(N) = 1/(N*(N - 1)), B(N) = B(N - 1) + A(N), B(0) = 0, and start with N = 1. Watch what happens to B(N) for the first few hundred values of N.

R. Taylor Series

1. PURPOSE AND DESCRIPTION

This program enables you to study polynomial approximations of functions of a single variable. The polynomials take the form

$$A_N(X - A)^N + \ . \ . \ . + A_1(X - A) + A_0$$

with $N \leq 12$. You enter a function Y = F(X), describe a graphing scale by giving minimum and maximum values for X and Y, choose the value of the expansion point A, and assign values to the coefficients A_N. You may then graph F(X), plot any of the partial sums, plot individual terms, and plot the differences between F(X) and the polynomial's partial sums. The various graphing options allow you to see the effects of coefficient changes.

2. STEP BY STEP

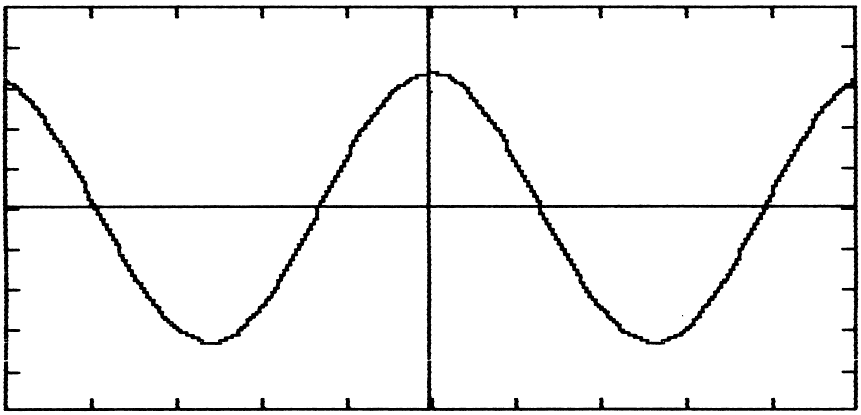
Load the program from the disk menu, read the greeting message, and go on to the input menu shown in Screen 1.

FUNCTION	
F(X) = COS(X)	
XY-REGION	COEFFICIENTS
XMIN = -6	A(0) = 1
	A(1) = 0
XMAX = 6	A(2) = -.5
	A(3) = 0
YMIN = -1.5	A(4) = .041666
	A(5) = 0
YMAX = 1.5	A(6) = -1.389E-03
	A(7) = 0
EXPANSION POINT	A(8) = 2.48E-05
	A(9) = 0
A = 0	A(10) = -2.756E-07
	A(11) = 0
	A(12) = 2.088E-09

|C|HANGE VALUES |G|O ON |Q|UIT

Screen 1. The input menu. After F has been graphed, the list of options shown here will be expanded automatically to include the option |L|AST GRAPH. Pressing |L| will then enable you to graph new polynomials without erasing old ones.

After reading the input menu, press |G| to plot F(X) = COS(X) over the interval -6 < X < 6 (Screen 2). Then, when you are ready, press |RETURN| to display the command menu, shown in Screen 3.



PRESS RETURN TO CONTINUE |_|

-6 < X < 6 -1.5 < Y < 1.5

XUNIT 1.2 YUNIT .3

Screen 2. The graphics screen, showing the current function Y = COS(X).

```
F(X) = COS(X)

GRAPHING COMMANDS                    -6 < X < 6
                                     -1.5 < Y < 1.5
|NP| NEW PLOT (ERASES                   A = 0
     SCREEN FIRST)
|PF| PLOT F(X)                       N   A(N)
|PP| PLOT PARTIAL SUM                0   1
|PT| PLOT TERM                       1   0
|PE| PLOT ERROR                      2   -.5
                                     3   0
     TO ERASE A PLOT                 4   .041666
     USE |E| FOR |P|                 5   0
                                     6   -1.389E-03
BRANCHING COMMANDS                   7   0
                                     8   2.48E-05
|S|  SWITCH: GRAPH/MENU              9   0
|I|  INPUT SCREEN                    10  -2.756E-07
|Q|  QUIT                            11  0
                                     12  2.088E-09
COMMAND? |_|
```

Screen 3. The command menu. You can enter coefficients and operate the program from either this screen or the next.

Operation commands are listed on the left side of the screen. Pressing the highlighted letter or letters will execute the commands. These commands can all be executed from the graphics screen as well. In particular, pressing |S| will exchange displays with a single keystroke. Try it.

The coefficients of the current approximating polynomial are listed down the right side of Screen 3. The current values are the coefficients of the Taylor polynomial

$$S_{12}(X) = 1 - \frac{X^2}{2!} + \frac{X^4}{4!} - \frac{X^6}{6!} + \frac{X^8}{8!} - \frac{X^{10}}{10!} + \frac{X^{12}}{12!}.$$

This is the 12th degree partial sum of the Taylor series expansion of F(X) = COS(X) about X = 0. We shall describe how to enter and change coefficient values later in the demonstration.

Example 1. <u>Investigating the Taylor polynomial approximations of F(X) = COS(X) near X = 0</u>.

The function and the coefficients of the approximating sums from $S_0(X)$ to $S_{12}(X)$ are the ones that appear in the opening views of the input screen and command menu (Screens 1 and 3).

Press |P| |P|, and when the prompt

PARTIAL SUM THRU TERM? |_|

appears, press |6| |RETURN|. The graph of

$$S_6(X) = 1 - \frac{X^2}{2} + \frac{X^4}{4!} - \frac{X^6}{6!}$$

will be added to the screen.

Now press |P| |T|, and when the prompt

TERM NUMBER? |_|

appears, press |8| |RETURN| to add the graph of the term

$$A(8)X^8 = \frac{X^8}{8!}$$

to the display.

To see the effect of adding this term to $S_6(X)$, plot $S_8(X)$ by pressing |P| |P| and then |8| |RETURN|.

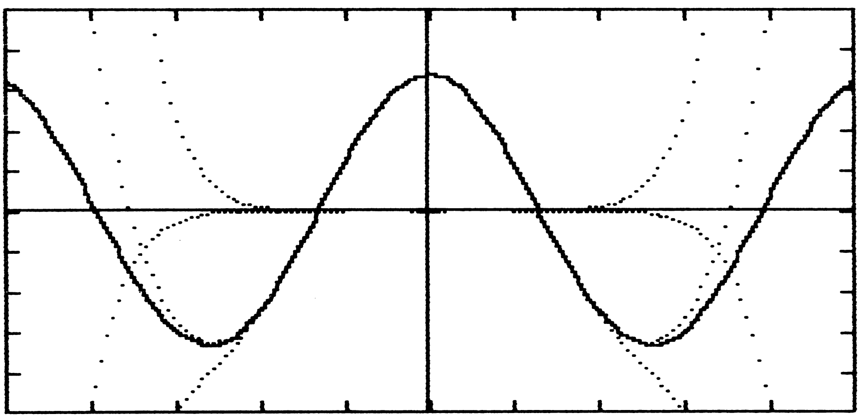
To plot the error

$$E_8(X) = COS(X) - S_8(X)$$

press |P| |E|, and when the prompt

ERROR THRU TERM? |_|

appears, press |8| |RETURN|. The graph of $E_8(X)$ will be added to the screen. The display should now look like the one in Screen 4.



COMMAND? |_|
-6 < X < 6 -1.5 < Y < 1.5
XUNIT 1.2 YUNIT .3

Screen 4. The graphs of COS(X), $S_6(X)$, $A(8)X^8 = X^8/8!$, $S_8(X) = S_6(X) + X^8/8!$, and $E_8(X) = COS(X) - S_8(X)$.

Any plot can be erased from the screen by replacing the |P| by |E| in the plot command. To erase $E_8(X)$, for example,

press |E| |E| and then press |8| |RETURN| when the prompt ERROR THRU TERM? |_| appears.

Example 2. Entering coefficients to build a new polynomial.
F(X) = SIN(X).

Press |I| to call up the input screen. Then press |C|, enter F(X) = SIN(X), and press five |RETURN|s to accept the current XY-region and expansion point. This will leave the cursor blinking at the value of A(0).

To construct the Taylor polynomial

$$S_{11}(X) = X - \frac{X^3}{3!} + \frac{X^5}{5!} - \frac{X^7}{7!} + \frac{X^9}{9!} - \frac{X^{11}}{11!}$$

at this point you could enter the coefficients

$$A(1) = 1, \quad A(3) = -1/3!, \quad \ldots, \quad A(11) = -1/11!$$

by keying in their numerical values one at a time along with zeros for the coefficients of even index. It is quicker, however, to enter

A(1) = 1,
A(3) = -A(1)/6,
A(5) = -A(3)/20,
A(7) = -A(5)/42,
A(9) = -A(7)/72,
A(11) = -A(9)/110.

When all the coefficients have been entered, the screen should show the following values:

A(0) = 0	A(7) = -1.9841E-04
A(1) = 1	A(8) = 0
A(2) = 0	A(9) = 2.7557E-06
A(3) = -.166666667	A(10) = 0
A(4) = 0	A(11) = -2.5052E-08
A(5) = 8.3333E-03	A(12) = 0.
A(6) = 0	

Once a coefficient has been entered, you may define another coefficient in terms of it in any way you please. You could have entered A(3) = -.167 and then defined A(1) = -6*A(3). Or you could have defined A(5) as

$$A(5) = (A(1) + A(3))/100.$$

With A(0) = 0, you could even have defined A(1) by the formula

$$A(1) = COS(A(0)).$$

Press |G| to plot F(X) = SIN(X). Then graph an assortment of partial sums and the error terms associated with them. When you have finished, press |I| to return to the input screen for the next example.

Example 3. Expanding SIN(X) about X = PI.

The general formula for the Taylor series expansion of a function F(X) about the point X = A is

$$F(X) = F(A) + F'(A)(X-A) + \frac{F''(A)}{2!}(X-A)^2 + \ldots$$

$$+ \frac{F^{(N)}(A)}{N!}(X-A)^N + \ldots$$

When F(X) = SIN(X) and A = PI, the first six coefficients are

N	$F^{(N)}$(PI)	A(N) = $F^{(N)}$(PI)/N!
0	SIN(PI) = 0	0
1	COS(PI) = -1	-1
2	-SIN(PI) = 0	0
3	-COS(PI) = 1	1/3!
4	SIN(PI) = 0	0
5	COS(PI) = -1	-1/5!

The Taylor polynomial $S_5(X)$ is therefore

$$S_5(X) = -(X - PI) + \frac{1}{3!}(X - PI)^3 - \frac{1}{5!}(X - PI)^5.$$

Press |C| on the input screen, followed by five |RETURN|s to accept F(X) = SIN(X) and the current XY-region. Then press |P| |I| |RETURN| to enter A = PI as the new expansion point.

Next, enter the coefficients of S_5(X). When you have finished, the coefficient column should display zeros except for

A(1) = -1, A(3) = .166666667, A(5) = -8.3333E-03.

Press |G| to graph F(X) = SIN(X). Then press |P| |P| and |1| |RETURN|, |P| |P| and |3| |RETURN|, and |P| |P| and |5| |RETURN|, in turn, to add the graphs of S_1(X), S_3(X), and S_5(X) to the display.

Now press |I| and enter the coefficients

A(7) = -A(5)/42, A(9) = -A(7)/72, A(11) = -A(9)/110.

After checking your entries, press |L| to recall the graphics display. Then press |P| |P| |1| |1| |RETURN| to add the graph of the partial sum S_{11}(X) to the display. The final picture should be the one in Screen 5.

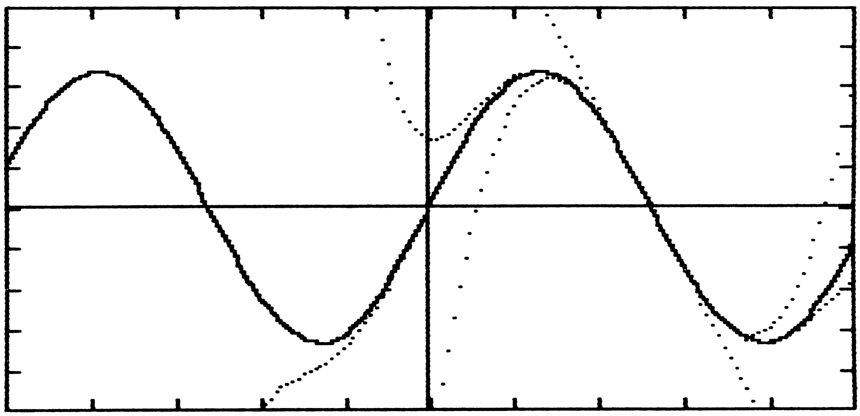
To get a closer view of the partial sum approximation near A = PI, press |I| and enter

XMIN = 0 XMAX = 3 YMIN = -2 YMAX = 2

Then replot F and the partial sums S_1, S_3, and S_5.

3. OTHER FEATURES

After investigating one approximating polynomial, you may wish to experiment with another while keeping your earlier graphs. To do so, enter the new polynomial's coefficients on the input menu, press |L| for last graph, and add the new graph for comparison.



COMMAND? |_|
-6 < X < 6 -1.5 < Y < 1.5
XUNIT 1.2 YUNIT .3

Screen 5. The graph of SIN(X) and the partial sums S_1(X), S_3(X), S_5(X), and S_{11}(X) of its Taylor series expansion about the point A = PI.

PROBLEMS

1. The series

$$X^2 - \frac{X^6}{3!} + \frac{X^{10}}{5!} - \frac{X^{14}}{7!} + \ . \ . \ .$$

converges to SIN(X^2) for all X. The convergence is especially rapid on the interval $-1 \leq X \leq 1$. Graph F(X) = SIN(X^2) and the partial sums for N = 2, 6, and 10 in a common graph for $-1 \leq X \leq 1$, $-2 \leq Y \leq 2$.

2. The coefficients A(N) of the geometric series

$$1 + X + X^2 + X^3 + \ldots + X^N + \ldots$$

are all equal to 1. Investigate the convergence of the series to the function F(X) = 1/(1 - X) by plotting F(X) together with the partial sums for N = 0, 1, 10, 11, and 12 over the interval $-2 \le X \le 1$. Use $-5 < Y < 10$.

3. The Taylor series expansion of the natural logarithm of X about A = 1 is

$$\text{LOG}(X) = (X-1) - \frac{(X-1)^2}{2} + \frac{(X-1)^3}{3} - \ldots$$
$$+ (-1)^{N+1}\frac{(X-1)^N}{N} + \ldots$$

The series converges for $0 < X \le 2$. Enter F(X) = LOG(X) and take $.1 \le X \le 4$, $-4 \le Y \le 4$, A = 1, and $A(N) = (-1)^{N+1}/N$ for N = 1, 2, ..., 12. Graph the partial sums for N = 1, 2, 3, 9, and 12. Note their erratic behavior for $X > 2$.

4. The Taylor series expansion of ATN(X) about X = 0 begins

$$\text{ATN}(X) = X - \frac{X^3}{3} + \frac{X^5}{5} - \frac{X^7}{7} + \frac{X^9}{9} - \frac{X^{11}}{11} + \ldots$$

The series converges for $-1 \le X \le 1$. To investigate the convergence of the partial sums to ATN(X), enter the function and graph it for $-3 \le X \le 3$, $-3 \le Y \le 3$. Then enter A(0) = 0, A(1) = 1, A(2) = 0, A(3) = -1/3, and so on up to A(11) = -1/11 and A(12) = 0, and graph the partial sums for N = 1, 2, 5, and 11. Plot the error of each partial sum.

5. The formula

$$\text{TAN}(X) \doteq X + \frac{X^3}{3} + \frac{2X^5}{15}$$

gives a close approximation for $-1 \leq X \leq 1$. Graph the function TAN(X), the polynomial, and the error in the approximation

a) for $-1.5 < X < 1.5$, $-10 < Y < 10$

b) for $-1 < X < 1$, $-1.5 < Y < 1.5$

6. The values of $\sqrt{1 + X^4}$ are sometimes calculated from the approximation

$$\sqrt{1 + X^4} \doteq 1 + \frac{X^4}{2} - \frac{X^8}{8}.$$

Graph the function and the polynomial together for $-1 \leq X \leq 1$. Try to improve the approximation by adding a multiple of X^{12} to the polynomial. [Hint: Plot the error in the approximation $S_8(X)$ and try to cancel it with a multiple of X^{12}.]

7. Try to make a good polynomial approximation of $F(X) = SEC(X)$ for $-1 \leq X \leq 1$. Work initially with $-1.5 < X < 1.5$ and $-1 < Y < 4$. Take $A(0) = 1$ and use only even powers of X. [The general rule is: To approximate an even function, use even powers of X or $(X - A)$; to approximate an odd function, use odd powers of X or $(X - A)$.]

S. Complex Number Calculator

1. PURPOSE AND DESCRIPTION

This program calculates sums, differences, products, quotients, and powers of complex numbers entered in either polar or rectangular form and shows the resulting Argand diagrams. The program also calculates the polar and rectangular forms quickly from one another.

2. STEP BY STEP

After loading the program, read the greeting message and continue on to the πcalculation padπ shown in Screen 1.

The pad has five rows, a row for each of the numbers in the equation

A |<OPERATION>| B = C

and two additional rows, labeled S and T, for storing results.

```
       REAL    IMAGINARY    MODULUS   ARGUMENT
    . . . . . . . . . . . . . . . . . . . . . . .
    A  0          0            0          0
   |+|
    B  0          0            0          0
    =
    C  0          0            0          0
    . . . . . . . . . . . . . . . . . . . . . . .
    S  0          0            0          0
    . . . . . . . . . . . . . . . . . . . . . . .
    T  0          0            0          0
    . . . . . . . . . . . . . . . . . . . . . . .

 CHANGE ENTRY:  |A| |O|PERATION |B| |S| |T| |C|LEAR
 |M|ODE:  RECTANGULAR |G|RAPH |Q|UIT | |
```

Screen 1. The calculation pad.

Example 1. Calculation with

$$A = 3 + 4i, \quad B = - 5 - i.$$

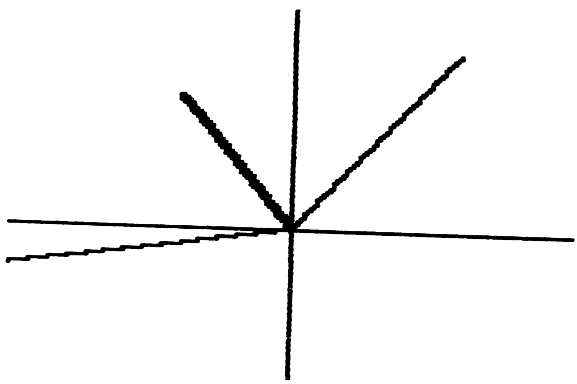
Enter A = 3 + 4i into the calculation pad by pressing |A| |3| |RETURN| |4| |RETURN|. After the second return, the computer will calculate and display the modulus and argument of A, which are 5 and .927295 respectively, and the cursor will move to the operation display. Press |RETURN| to accept the default operation, addition. Then press |-| |5| |RETURN| |-| |1| |RETURN| to enter B = −5 − i. The computer will complete the B row by displaying the modulus and argument of B = −5 − i, and will then calculate and display the rectangular and polar coordinates of the sum A + B = C. The pad's first three rows should now look like this:

	REAL	IMAGINARY	MODULUS	ARGUMENT
A	3	4	5	.927295
|+|				
B	−5	−1	5.09902	3.33899
=				
C	−2	3	3.60555	2.15880

Press |G| to construct the Argand diagram for

$$(3 + i) + (-5 - i) = -2 + 3i,$$

shown in Screen 2.



A, B, A + B SCALE: 5

PRESS RETURN TO CONTINUE |_|

Screen 2. The Argand diagram for A + B = C with A = 3 + 4i and B = −5 − i. The scale is determined by the coordinate of largest absolute value in the rectangular representations of A, B, and C.

After viewing the Argand diagram, press |RETURN| to continue the example.

Move the results of the addition into row S by pressing |S| and then, after a pause, |C|. Notice the change in

instructions at the bottom of the screen: Immediately after you press |S| the bottom line will change to

|RETURN| ACCEPT ENTRY |ESC| ABORT ENTRY
USE ENTRY IN: |A| |B| |C| |S| |T| |Z|ERO.

The options shown here enable you to enter the contents of any row into row S by pressing the letter of that row. The options even allow you to keep S as it is by pressing |S| itself (useful if you happen to have pressed |S| by mistake and have no desire to change the row), or to enter zeros by pressing |Z| (useful for clearing a row).

After pressing |S| and |C| to duplicate row C in row S, the cursor will move to the first entry in row T. Press |ESC| to exit from the T row.

Now press |O| (πoh,π not zero), |/|, and |ESC| to calculate

$$A/B = (3 + 4i)/(-5 - i) = -.730769 - .653846\, i.$$

Press |G| to see the Argand diagram of the quotient, and, when ready, press |RETURN| to begin the next example.

Example 2. Calculations with

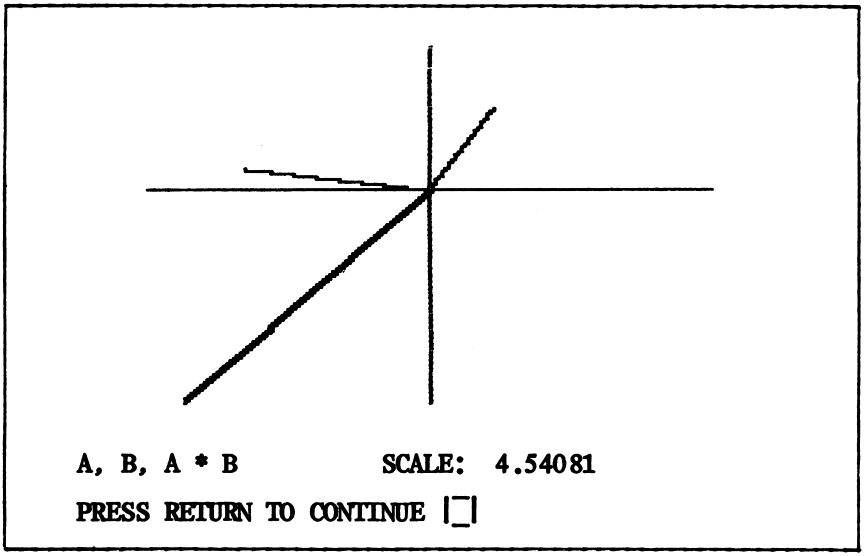
$$A = 2e^{i} \quad \text{and} \quad B = 3e^{3i}.$$

Press |C| to clear the calculation pad. The entries will then be all zeros. Press |M| to change from rectangular to polar mode. Then press |A| |2| |RETURN| |1| |RETURN| to enter $A = 2e^{i}$. Press |*| to request multiplication, and press |3| |RETURN| |3| |RETURN| to enter $B = 3e^{3i}$. Row C will then display the product of A and B in both rectangular and polar form. To five decimal places,

$$2e^{i} * 3e^{3i} = -3.92186 - 4.54081\, i = 6e^{4i}.$$

Press |G| for the Argand diagram, shown in Screen 3. Notice how the scale in this case is determined by the magnitude of the i-component of the rectangular form of the product. When you return to the calculation pad, you will

see that once again the scale has been determined by the coordinate of the largest magnitude in the rectangular representations of A, B, and C.



Screen 3. Argand diagram for $2e^{i} * 3e^{3i} = 6e^{4i}$.

Press |RETURN| to return to the calculation pad.

Other algebraic combinations of A and B are calculated by pressing |O| |<OPERATION SYMBOL>| |ESC|. To calculate B - A, B/A, and B^A, however, you must first reverse the present order of A and B by interchanging the contents of the A and B registers. To do this you can move A to S (for temporary storage), B to A, and S to B.

3. SPECIAL INPUTS

The numbers π and e can be entered in numerical and functional expressions as PI and E. Press |P| |I| for π, and |E| for e.

PROBLEMS

In Problems 1–8, calculate A + B, A − B, A*A, A*B, B*B, 1/A, 1/B, A/B, B/A, A^B, A^{-B}, B^A, and B^{-A}. (Toolkit notation for A^B is A^B.)

1. $A = 2 + 3i$, $B = 4 - 2i$
2. $A = 2 - i$, $B = -2 + 3i$
3. $A = -1 - 2i$, $B = .5$
4. $A = 1 + i$, $B = 1 - i$
5. $A = 5$, $B = 3 - 4i$
6. $A = 2e^{i}$, $B = 3e^{3i}$
7. $A = e^{\pi i}$, $B = -1$
8. $A = i$, $B = i$

9. Calculate $(1 + i)^{(1+i)}$ algebraically. Then check your result with COMPLEX NUMBER CALCULATOR.

T. 3D Grapher

1. PURPOSE

This program graphs surfaces defined by equations of the form

$$z = f(x,y)$$

for (x,y) in a rectangular domain.

2. DESCRIPTION

3D GRAPHER produces 2-dimensional images of surfaces defined in a 3-dimensional rectangular coordinate system. The screen shows an image of the surface projected by rays from a viewing point P through the surface S to a plane behind it, as in Figure 1.

The spherical coordinates of the viewing point determine the perspective. The distance D of the image plane from the viewing point is also controllable. Together, the choice of spherical coordinates and the value of D determine the image size. You enter F(X,Y), the maximum and minimum X and Y

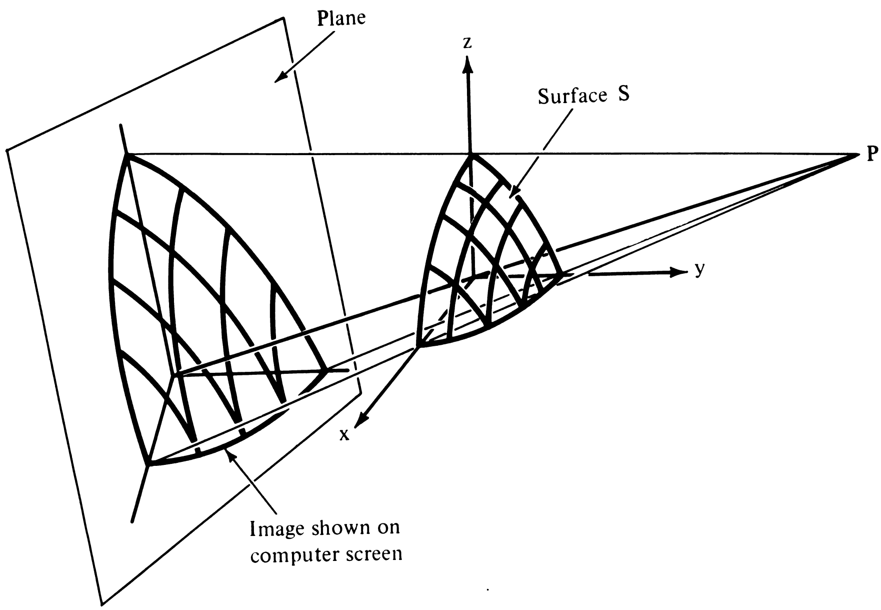


Figure 1. The projection of a surface S onto an image plane located D units away, and opposite the origin, from the viewing point P.

values, spherical coordinates for the viewing point, the value of D, and the number of X values and Y values to plot.

The surface may be represented by plane sections parallel to the yz-plane, by plane sections parallel to the xz-plane, by both together, or by a collection of unconnected points on the surface. Thus, the surface is presented as a thin, transparent shell that may, at your option, be made visible by hatching, crosshatching, or dotting. Figures 2-5 illustrate the different options.

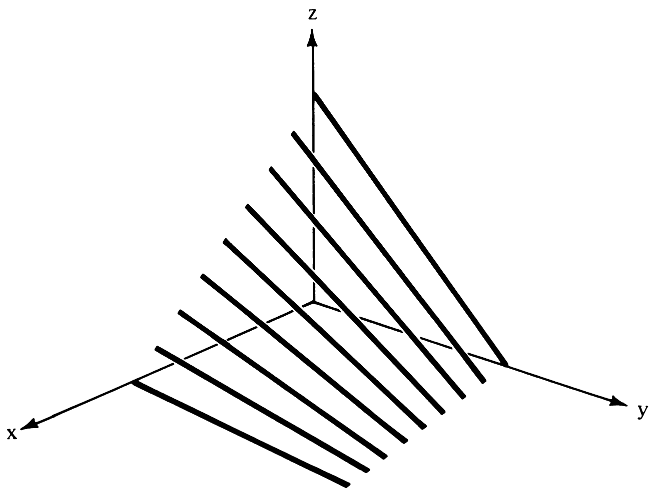


Figure 2. The surface $z = (x - 1)(y - 1)$ sketched over the rectangle $0 \leq x \leq 1$, $0 \leq y \leq 1$ by lines parallel to the yz-plane.

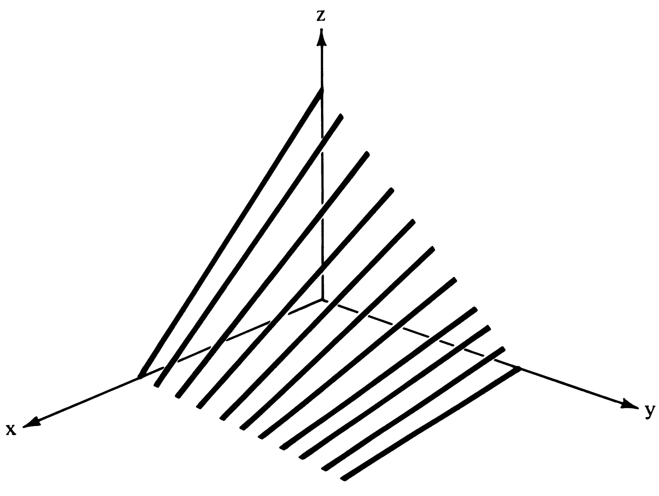


Figure 3. The surface $z = (x - 1)(y - 1)$ sketched over the rectangle $0 \leq x \leq 1$, $0 \leq y \leq 1$ by lines parallel to the xz-plane.

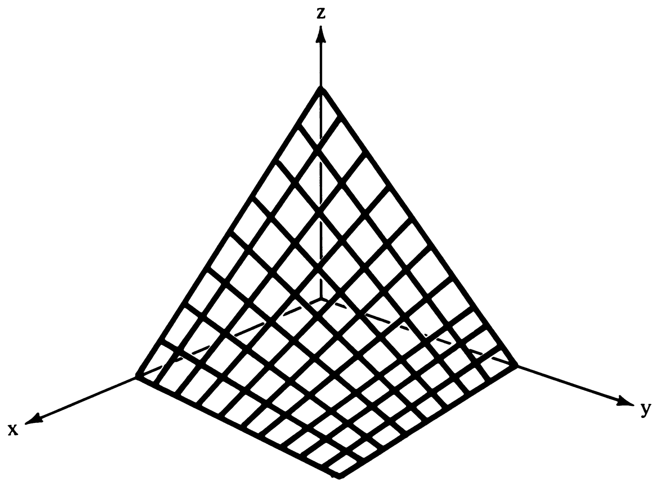


Figure 4. The surface $z = (x - 1)(y - 1)$ sketched over the rectangle $0 \leq x \leq 1$, $0 \leq y \leq 1$ by crosshatching.

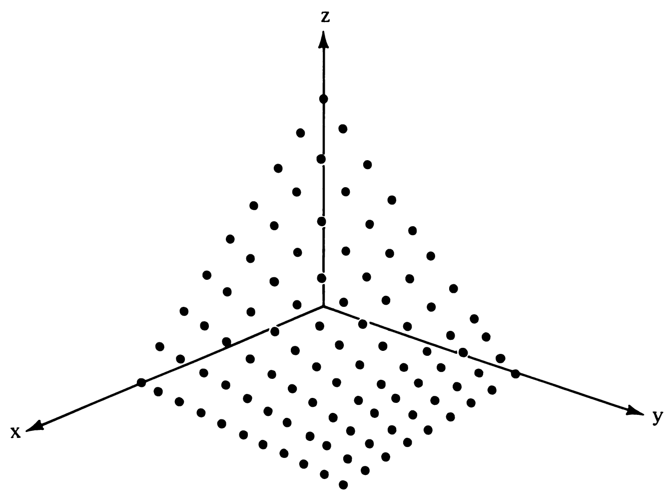


Figure 5. The surface $z = (x - 1)(y - 1)$ over the rectangle $0 \leq x \leq 1$, $0 \leq y \leq 1$, revealed by dotting.

The point P is located by its spherical coordinates (ρ,θ,ϕ), as shown in Figure 6. The image plane is located opposite the origin from P, and at distance D from P. In order to have this relationship, D must be greater than ρ. In addition, the variable ϕ is required to satisfy $0 < \phi < \pi$ to avoid potentially confusing interpretations.

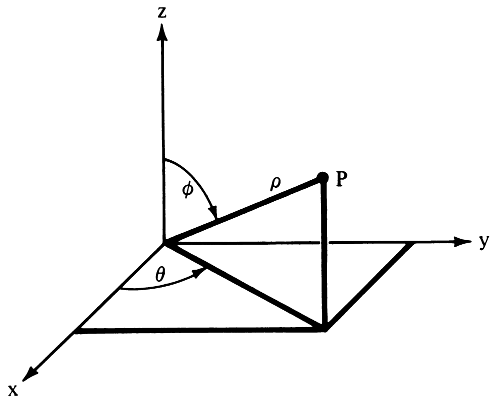


Figure 6. The spherical coordinates of the viewing point P.

The point P may also be regarded as the location of an observer's eye. When the location of P is changed, the viewer sees S from a new perspective, and the size of the image may change. In particular, a decrease in the value of ρ produces an increase in image size and some distortion as well. In fact, if ρ is decreased enough so that P moves in too close to S or even crosses it, the image may become unrecognizable, or it may be lost altogether.

Thus, while at times it may be desirable to view the surface from up close, it is often better to reduce distortion by using values of ρ that are relatively large compared with the values of x, y, and z for points in S. However, increasing ρ decreases the image size, so the image plane must be moved <u>away</u> by increasing D to maintain good presentation.

The size of the image is determined by the ratio D/ρ and the ranges of values assumed by the variables x, y, and z. Some experimenting will be necessary to develop a sense of how to choose values of D and ρ to achieve images of desired size. For some rough guidance, we note that the choice $D/\rho = 60$ produces a good image of the surface in Example 2, Screen 4, in which the values of x, y, and z are all in the interval from 0 to 1, with each attaining a maximum of 1. The value 40 is reasonable for Example 3, Screen 6, in which x and y vary from 0 to 2 and z varies from -1 to 1.

The variables Θ and $\emptyset$ are also important in obtaining good images. Features of a surface hidden from one viewpoint may be revealed clearly from another. It is often necessary to try several angles before obtaining a clear picture of a surface. The graphing subroutine in 3D GRAPHER is written in machine language to produce graphs quickly and to enable you to experiment without spending a lot of time.

The final parameters that affect the display are the number of x- and y-curves for hatching or crosshatching. If the image size is small, these numbers should be correspondingly small. These two numbers also determine the number of dots plotted when the surface is made visible by dots rather than hatching. In this case, the number of dots should usually be taken large for good presentation. In using these numbers, the count starts from 0 instead of 1 so there will be one more curve in each direction than the number specified.

3. STEP BY STEP

Load the program from the disk menu, read the greeting messages, and continue on to the program menu, shown in Screen 1.

```
                    PROGRAM MENU
         |P| .. PROBLEM DISPLAY
                  ... OR PLOT SURFACE USING
         |X| .. X-CURVES
         |Y| .. Y-CURVES
         |B| .. BOTH X-CURVES AND Y-CURVES
         |D| .. DOT MODE
                  ... OR EXIT USING
         |Q| .. QUIT

         PRESS LETTER OF YOUR CHOICE
```

Screen 1. The program menu.

```
EQUATION AND ENDPOINTS

F(X,Y) = X * X - Y * Y

  XMIN = -1          XMAX = 1
  YMIN = -1          YMAX = 1

VIEWING PARAMETERS

   RHO = 15         THETA = 1
   PHI = 1.2        DIST. = 800

NUMBER OF X,Y VALUES FOR PLOTTING

     N = 10            M = 10

|C|HANGE ENTRY   |G|O ON   |Q|UIT   |_|
```

Screen 2. The problem menu with entries from Example 1.

Example 1. Graph the surface

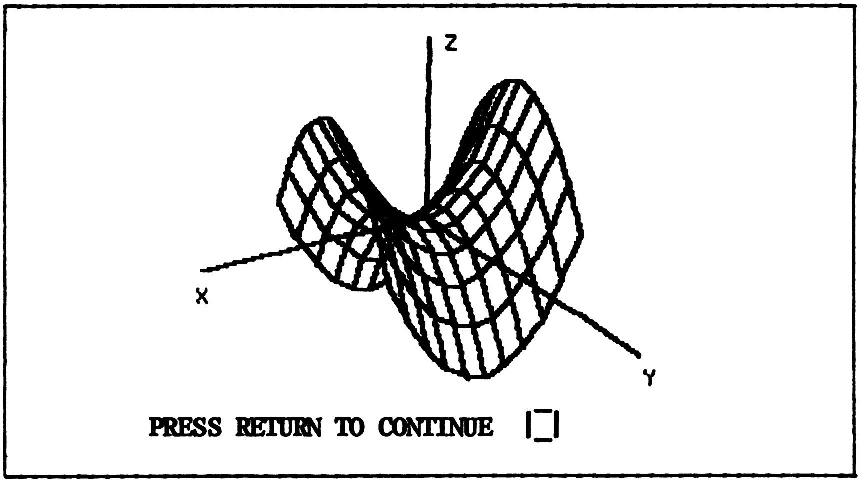
$$z = x^2 - y^2$$

for $-1 \leq x \leq 1$, $-1 \leq y \leq 1$, using $\rho = 15$, $\theta = 1$, $\phi = 1.2$, $D = 800$. Use $n = 10$ x-curves and $m = 10$ y-curves.

Solution. Starting from the program menu (Screen 1), press |P| to see how the surface equation and plotting parameters for this example have been entered into the problem menu (Screen 2).

Note the computer variable names assigned to the viewing parameters: RHO, THETA, PHI, DIST. (for D), etc. Press |G| to accept all entries.

You are now back at the program menu. Press |B| for the graph. The computer will draw the x-curves first, then the y-curves, to give the display in Screen 3.



Screen 3. The hyperbolic paraboloid $z = x^2 - y^2$ graphed over the square $|x| \leq 1$, $|y| \leq 1$.

Press |RETURN| to return to the program menu, which now has the option

|L| .. LAST SURFACE DISPLAYED.

Pressing |L| at this point will recall the plot shown in Screen 3. Try it.

Press |RETURN| to continue, then press |X| to see the paraboloid represented by x-curves only. Then press |RETURN| and |Y| to see it drawn with y-curves only.

Example 2. Graph the surface

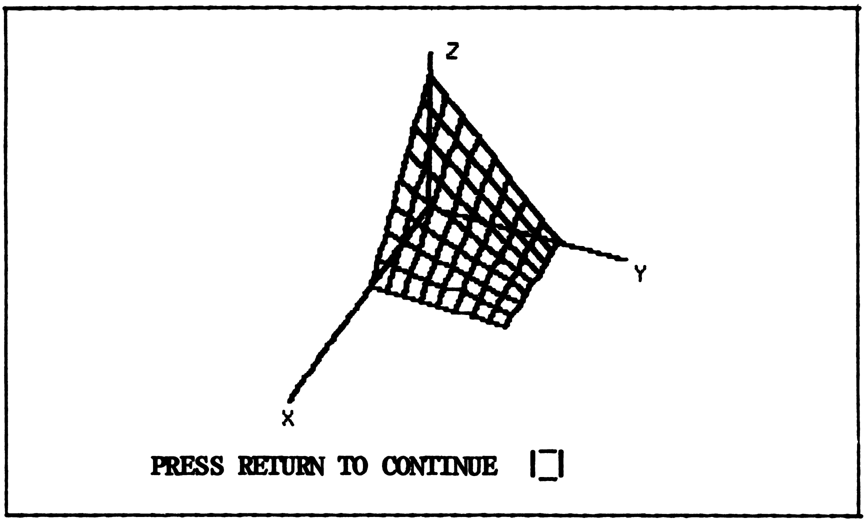
$$z = (x - 1)(y - 1)$$

for $0 \leq x \leq 1$, $0 \leq y \leq 1$. Use $\rho = 15$, $\theta = .4$, $\phi = 1$, $D = 1000$, and $N = M = 8$. Also, plot the surface using the dot mode, with 20 dots in each direction.

Solution. From the program menu, press |P| then |C| to enter the formula

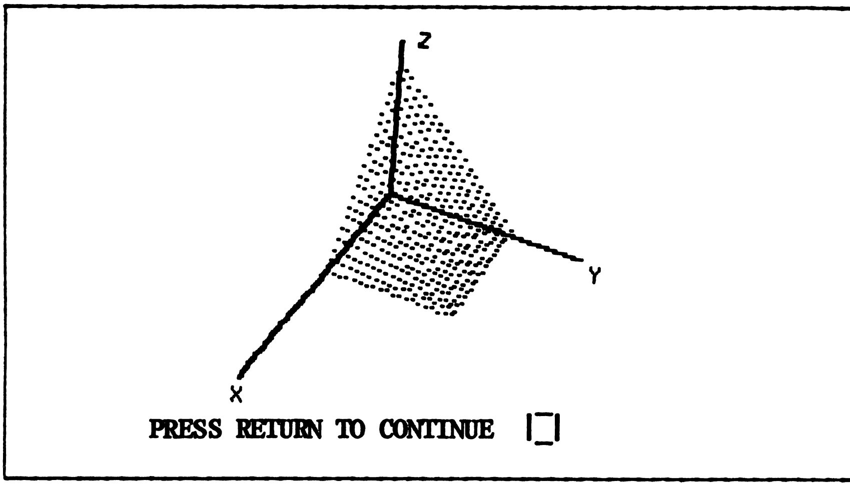
F(X,Y) = (X - 1) * (Y - 1)

and the new parameter values. Then press |G| and |B| to obtain the graph shown in Screen 4.



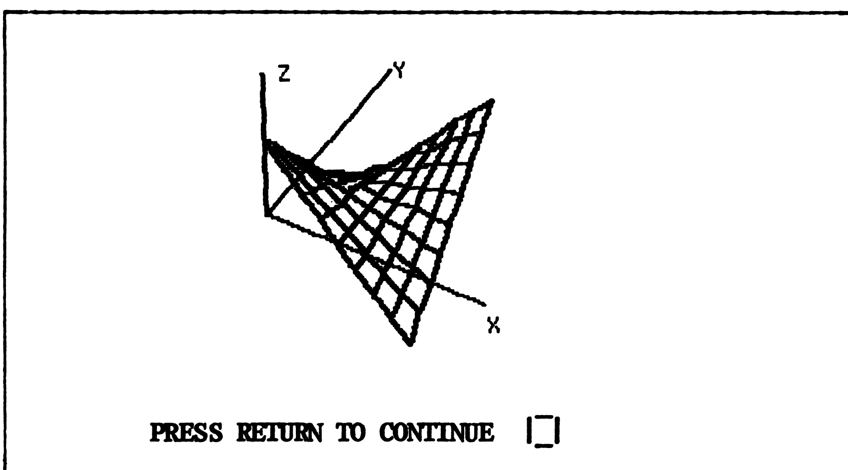
Screen 4. The surface $z = (x - 1)(y - 1)$, crosshatched over the unit square $0 \leq x \leq 1$, $0 \leq y \leq 1$.

Now return to the program menu, press |P|, then |C|, and enter M = N = 20. Then press |G| and |D| for a dotted graph (Screen 5).



Screen 5. A dot-mode graph of the same surface.

Example 3. Graph the surface of Example 2 over the extended domain $0 \leq x \leq 2$, $0 \leq y \leq 2$. Use $\rho = 40$, $\theta = -1$, $\phi = .9$, $D = 1600$, and $N = M = 8$.



Screen 6. The graph of $z = (x - 1)(y - 1)$, over the rectangle $0 \leq x \leq 2$, $0 \leq y \leq 2$ shows a saddle.

Solution. From the program menu press |P|, then press |C| and make the necessary entries. Then press |G| and |B| for the graph, shown in Screen 6.

Examples 2 and 3 illustrate that some care must be taken to ensure an accurate representation. In Example 3 an extension of the plotting domain revealed a "saddle" that was not visible in Screen 5. In some cases, algebraic properties of the formula for a surface may give an indication of where to explore. For instance, the factors $x - 1$ and $y - 1$ in $z = (x - 1)(y - 1)$ indicate that the surface may have interesting features near the point (1,1). It is also easy to observe that along the line $y = x$ we have $y - 1 = x - 1$, so the formula for z along this line reduces to $(x - 1)^2$. This simple observation shows that the surface has a parabolic character not apparent in the original graph, and hence that further exploration is warranted.

Example 4. Graph the surface

$$z = 1 - x^2/4 - y^2$$

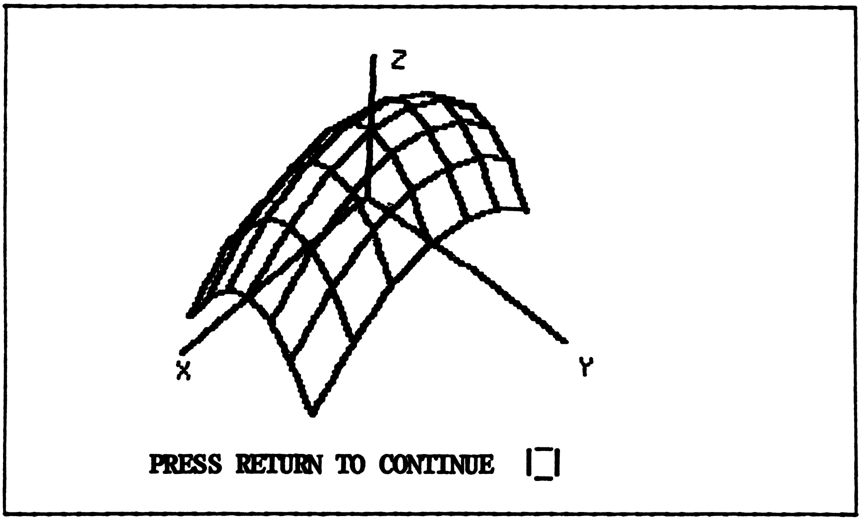
for $-2 \leq x \leq 2$, $-1 \leq y \leq 1$. Use $\rho = 20$, $\Theta = .7$, $\phi = .8$, $D = 700$, and $N = M = 6$.

Solution. Press |P| on the program menu and enter the surface equation

F(X,Y) = 1 - .25 * X * X - Y * Y

and the appropriate parameter values. Then press |G| and |B| for the graph. The result, shown in Screen 7, is an image of an elliptic paraboloid opening downward from its vertex, (0,0,1).

The x- and y-curves shown in Screen 7 are easily identified as parabolas. For example, by setting $x = c$, a constant, we find the relation $z = (1 - c^2/4) - y^2$, which is the equation of a parabola in y and z. And, while they are not shown explicitly in the display, we note that the z-curves are ellipses of the form $x^2/4 + y^2 = \text{const.}$



Screen 7. The elliptic paraboloid $z = x^2/4 - y^2$ over the rectangle $-2 \leq x \leq 2$, $-1 \leq y \leq 1$.

Thus, the computer display, coupled with some thought about the algebraic properties of the surface equation, should give a good idea of what the surface really looks like.

4. OVERCOMING OCCASIONAL DIFFICULTIES

The subroutine for masking portions of the surface that are hidden from view by other areas of the surface may not screen the "hidden lines" properly, especially when graphing the y-curves. This difficulty usually can be circumvented by changing the perspective on the surface.

There is also a sporadic failure of the curves in one direction to connect properly to the final curve in the other direction to complete the crosshatching of a surface. This can usually be handled by changing the number of curves taken for the crosshatching (e.g., use 7 or 8 instead of 6, or vice versa, for the number of x-curves and/or y-curves).

See Problem 13 for an instructive example.

5. |ESC|

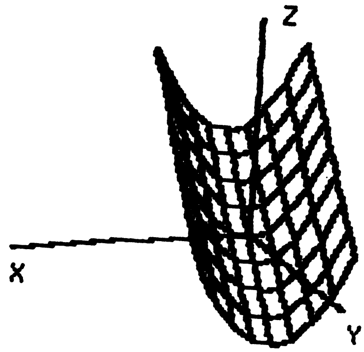
After pressing |C| to make changes in the problem menu (Screen 3), you can exit from change mode at any time by pressing |ESC|.

This can save time if you want to change a single entry early in the list. Press |C|, key in the new entry, press |RETURN|, and press |ESC|. The cursor will return directly to the bottom of the screen once |ESC| is pressed.

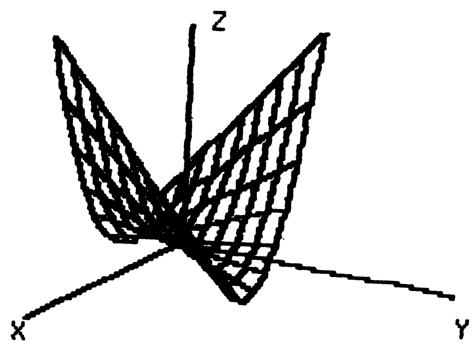
PROBLEMS

Graph the following surfaces, using the given parameter values. Compare your results with the pictures shown here.

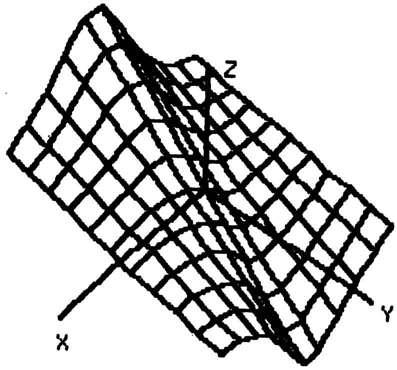
1. $z = x^2 - y$; $-1 \leq x \leq 1$,
   $-1 \leq y \leq 1$, $\rho = 30$,
   $\theta = 1.2$, $\phi = 1.3$,
   $D = 1000$, $n = 8$, $m = 8$.



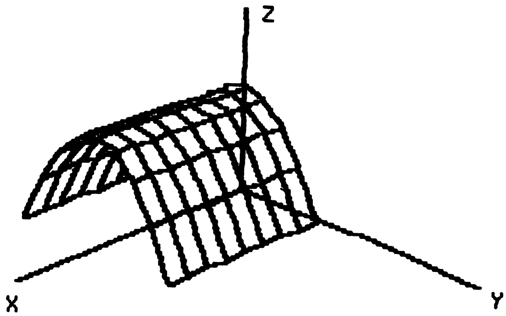
2. $z = x^2 - xy$; $-1 \leq x \leq 1$,
   $-1 \leq y \leq 1$, $\rho = 25$,
   $\theta = .5$, $\phi = 1.3$,
   $D = 1000$, $n = 9$, $m = 9$.



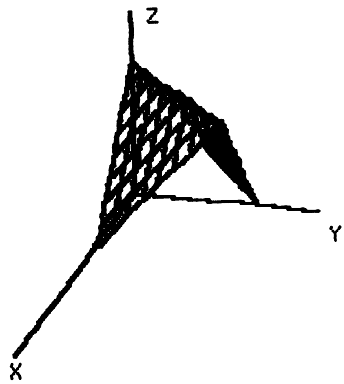
3. $z = (x - y)/(1 + x^2)$; $-2 \leq x \leq 2$, $-2 \leq y < 2$, $\rho = 20$, $\theta = .7$, $\phi = .8$, $D = 650$, $n = 10$, $m = 10$.



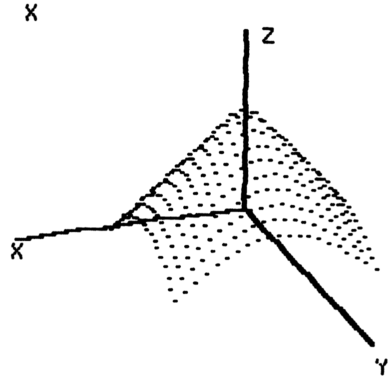
4. $z = 1 - y^2$; $0 \leq x \leq 2$, $-1 \leq y \leq 1$, $\rho = 25$, $\theta = .75$, $\phi = 1.2$, $D = 1200$, $n = 8$, $m = 8$.



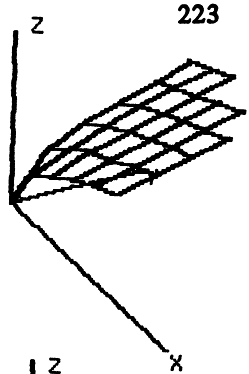
5. $z = 1 - |x - y|$; $0 \leq x \leq 1$, $0 \leq y \leq 1$, $\rho = 25$, $\theta = .3$, $\phi = 1.2$, $D = 1500$, $n = 8$, $m = 8$.



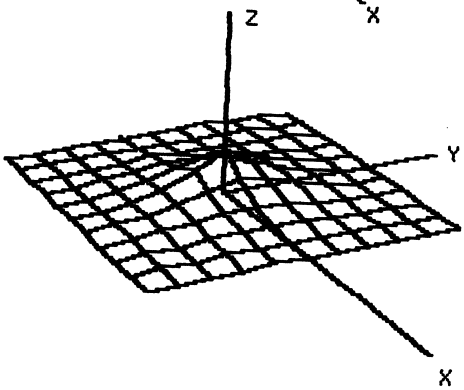
6. $z = 1 - \sqrt{x^2 + y^2}$; $-1 \leq x \leq 1$, $-1 \leq y \leq 1$, $\rho = 20$, $\theta = 1.2$, $\phi = 1.2$, $D = 1000$, $n = 20$, $m = 20$; use dot mode.



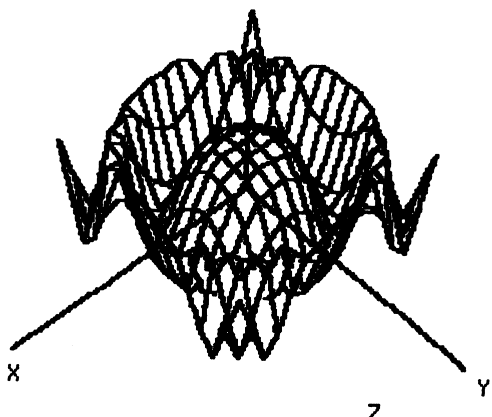
7. $z = \sqrt{x + y}$; $0 \leq x \leq 4$, $0 \leq y \leq 4$, $\rho = 50$, $\theta = -.5$, $\phi = 1.1$, $D = 1200$, $n = 5$, $m = 5$.



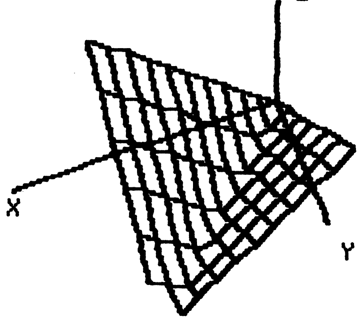
8. $z = 1/(1 + x^2 + y^2)$; $-4 \leq x \leq 4$, $-4 \leq y < 4$, $\rho = 50$, $\theta = -.5$, $\phi = 1.2$, $D = 1000$, $n = 10$, $m = 10$.



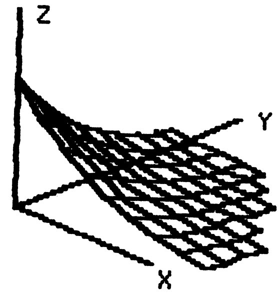
9. $z = \cos(x^2 + y^2)$; $-2.5 \leq x \leq 2.5$, $-2.5 \leq y \leq 2.5$, $\rho = 40$, $\theta = .8$, $\phi = .8$, $D = 1000$, $n = 15$, $m = 15$.



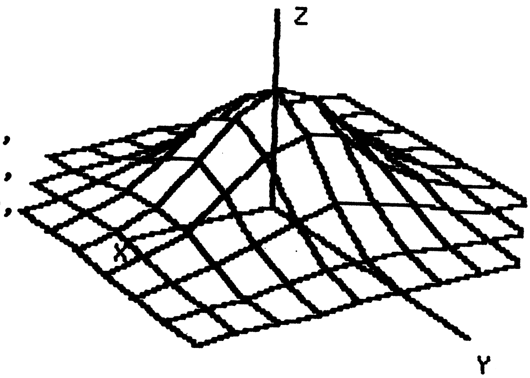
10. $z = |x - y|$; $0 \leq x \leq 5$, $0 \leq y \leq 5$, $\rho = 50$, $\theta = 1.2$, $\phi = .6$, $D = 800$, $n = 10$, $m = 6$.



11. $z = e^{-(x+y)}$; $0 \leq x \leq 2$, $0 \leq y \leq 2$, $\rho = 15$, $\theta = -.8$, $\phi = 1.2$, $D = 800$, $n = 8$, $m = 8$.



12. $z = e^{-(x^2+y^2)}$, $-2 \leq x \leq 2$, $-2 \leq y \leq 2$, $\rho = 30$, $\theta = 1$, $\phi = 1.2$, $D = 1500$, $n = 10$, $m = 8$.



13. To appreciate the improvement that can sometimes be achieved by changing the settings in the problem menu, try graphing $z = \sin(xy)$ with each of the following choices of settings:

a) $-4 \leq x \leq 4$, $-4 \leq y \leq 4$, $\rho = 15$, $\theta = 1$, $\phi = 1.2$, $D = 800$, $n = m = 10$
b) $-4 \leq x \leq 4$, $-4 \leq y \leq 4$, $\rho = 20$, $\theta = 1$, $\phi = .8$, $D = 600$, $n = m = 24$
c) $-6 \leq x \leq 6$, $-6 \leq y \leq 6$, $\rho = 40$, $\theta = 1$, $\phi = .8$, $D = 600$, $n = m = 24$

U. Double Integral Evaluator

1. DESCRIPTION

This program evaluates integrals of the form

$$\int_a^b \int_{g(x)}^{h(x)} f(x,y)\,dy\,dx$$

by an iterated trapezoidal rule, an iterated Simpson's rule, and an iterated Romberg method, and enables you to compare the three results. You key in formulas for f, g, and h, values for a and b, the number of x- and y-subintervals for the trapezoidal and Simpson rules, and an error tolerance for the Romberg integration.

2. THE NUMERICAL METHODS

The iterated trapezoidal rule, iterated Simpson's rule, and iterated Romberg integration are straightforward extensions of the methods used in the program INTEGRAL EVALUATOR described in Chapter M.

Under the iterated trapezoidal rule, the x-interval from a to b is first partitioned into n subintervals of equal length. The length of each subinterval is $h = (b - a)/n$, and the partition points are given by

$$x_i = a + ih, \quad i = 0, 1, \ldots, n.$$

Then for each i the y-interval from $g(x_i)$ to $h(x_i)$ is

partitioned into m subintervals, each of length

$$k_i = (h(x_i) - g(x_i))/m,$$

to obtain a "rectangular" grid on the region of integration, as shown in Fig. 1.

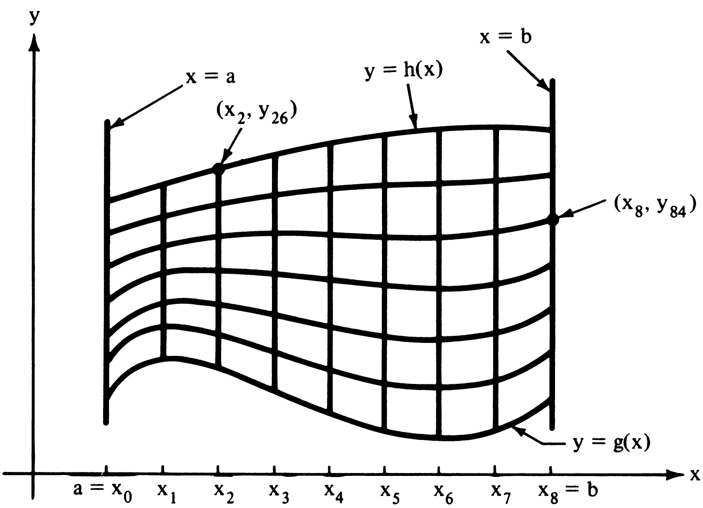


Figure 1. A sample partition of the region in the xy-plane bounded by the curves $y = g(x)$, $y = h(x)$, and the lines $x = a$, $x = b$, with $n = 8$, $m = 6$.

Denote the grid points by

$$(x_i, y_{ij}); \quad i = 0, 1, \ldots, n, \; j = 0, 1, \ldots, m,$$

and let

$$z_{ij} = f(x_i, y_{ij}); \quad i = 0, 1, \ldots, n, \; j = 0, 1, \ldots, m.$$

Then for each i form the trapezoidal sum

$$I_i = \frac{k_i}{2}(z_{i0} + 2z_{i1} + 2z_{i2} + 2z_{i,m-1} + z_{im}),$$

so that I_i is the mth trapezoidal approximation to the <u>inner integral</u>

$$\int_{g(x_i)}^{h(x_i)} f(x_i,y)\,dy.$$

The resulting approximation of the double integral is

$$T_{nm} = \frac{h}{2}(I_0 + 2I_1 + 2I_2 + \ldots + 2I_{n-1} + I_n).$$

For the iterated Simpson's rule, n and m must be <u>even</u> positive integers. The symbols h, x_i, y_{ij}, z_{ij}, k_i have the same meaning that they do in the iterated trapezoidal rule. For each i from 0 to n form the sum

$$I_i = \frac{k_i}{3}(z_{i0} + 4z_{i1} + 2z_{i2} + 4z_{i3} + \ldots + 2z_{i,m-2} + 4z_{i,m-1} + z_{im}),$$

so that I_i is the mth Simpson approximation to the inner integral

$$\int_{g(x_i)}^{h(x_i)} f(x_i,y)\,dy.$$

The resulting approximation of the double integral is

$$S_{nm} = \frac{h}{3}(I_0 + 4I_1 + 2I_2 + 4I_3 + \ldots + 2I_{n-2} + 4I_{n-1} + I_n).$$

Under the iterated Romberg method, the integration in each direction is carried out according to the Romberg scheme for single integrals. Thus, at the first stage the Romberg approximations

$$I_a \doteq \int_{g(a)}^{h(a)} f(a,y)\,dy \quad \text{and} \quad I_b \doteq \int_{g(b)}^{h(b)} f(b,y)\,dy$$

are used to form the first trapezoidal approximation in the x-direction,

$$T_1 = \frac{b-a}{2}(I_a + I_b).$$

This value is the first Romberg approximation to the double integral:

$$R_0 = T_1.$$

At the next stage, the interval a to b is divided in half, and the Romberg approximation to the inner integral is computed at the midpoint $c = (a + b)/2$:

$$I_c \doteq \int_{g(c)}^{h(c)} f(c,y)\,dy$$

The trapezoidal approximation

$$T_2 = \frac{b - a}{4}(I_a + 2I_c + I_b)$$

is then used to obtain the next Romberg approximation to the double integral:

$$R_1 = (4T_2 - T_1)/3.$$

The procedure is continued in the following way. At the ith stage the number of x-subintervals is again doubled, Romberg approximations are formed at each partition point, a trapezoidal approximation is formed, a row of extrapolations is carried out as described in Chapter M, and the final value is used as the ith Romberg approximation of the double integral. After computing a minimum of $i = 3$ rows, the procedure is halted when

$$|R_i = R_{i-1}| < \varepsilon |R_i|,$$

or when 7 rows are computed, whichever occurs first. The value 7 was chosen to preclude excessive run times; of course, this means that in some problems we may not get the accuracy we want.

3. STEP BY STEP

Load the program from the disk menu, read the greeting message, and continue on to the program menu, shown in Screen 1. Then work through the following examples.

PROGRAM MENU

|T| .. TRAPEZOIDAL RULE
|S| .. SIMPSON'S RULE
|R| .. ROMBERG INTEGRATION
|Q| .. QUIT

PRESS LETTER OF YOUR CHOICE |_|

Screen 1. The program menu.

Example 1. Evaluate

$$\int_0^1 \int_{1-x}^{1+x} (x^2 + y^2)\,dydx$$

by the three methods described above, and compare results. Use n = 10, m = 10 for both the trapezoidal and Simpson's rules. Let ε = .00001 for the Romberg integration.

Solution. Press |T| on the program menu to call up the problem display for the trapezoidal rule, shown in Screen 2.

Since the default problem in the program is correct for this example, press |G| to integrate. The trapezoidal approximations of the inner integrals at the eleven partition points are displayed as they are computed. You can halt the action by pressing |RETURN|, and continue it with another |RETURN|. The final trapezoidal approximation to the double integral, $T_{10,10}$, is shown as T = 1.6767.

The exact value of the integral is 5/3. Thus, T is accurate to only 2 places.

As noted above, the values I0 to I10 displayed on the left side of the screen are approximations of the inner integrals. For example, one of the values is I_5 = 1.335, which approximates the value $2(1/2) + 8(1/2)^3/3 = 4/3$ of the y-integral $2x + 8x^3/3$ at the partition point $x_5 = 1/2$.

```
TRAPEZOIDAL RULE INTEGRAND AND LIMITS
F(X,Y) = X * X + Y * Y
  G(X) = 1 - X
  H(X) = 1 + X
     A = 0              B = 1

NUMBER OF X-STEPS   NUMBER OF Y-STEPS
     N = 10             M = 10
|C|HANGE ENTRY   |G|O ON   |M|ENU   |Q|UIT   |_|
```

Screen 2. Problem display for the trapezoidal rule in Example 1.

Press |RETURN| to return to the program menu (Screen 1), which now contains the option

|H| .. HARD COPY OF LAST RESULTS.

If your computer is connected to a printer and you know the slot number of the interface card, you may obtain a copy of the values just calculated by pressing |H|, then the slot number, then |RETURN| to confirm your choice.

To continue the demonstration, press |S| from the program menu for Simpson's rule. The problem display will be nearly the same as the one in Screen 2. The input data should still be correct, so press |G| for the integration. The approximation $S_{10,10}$ is shown as S = 1.66666667. Press |RETURN| to continue; the final trapezoidal result is recalled and displayed for comparison.

Press |RETURN| to return to the program menu, and press |R| for Romberg integration. This time the problem display shows the error tolerance in place of the number of subintervals. Press |RETURN| then |G| for the integration. The value R = 1.66666667 is returned after computation of the minimum three extrapolation rows.

Press |RETURN| to display the results of all three computations (Screen 3).

```
F(X,Y) = X * X + Y * Y
  G(X) = 1 - X
  H(X) = 1 + X

A = 0            B = 1

R0 = 2.33333333        WITH N = 10
R1 = 1.66666667             M = 10
R2 = 1.66666667
                       T = 1.6767
 R = 1.66666667
                        WITH N = 10
                             M = 10

                       S = 1.66666667

PRESS ANY KEY TO CONTINUE |_|
```

Screen 3. The results of the three methods shown together.

The accuracy of the Simpson and Romberg approximations can be traced to the fact that the functions involved in the integrations in both directions are polynomials of degree no more than three. In fact, the correct eight-place value of the integral is returned by Simpson's rule even with $n = 2$ and $m = 2$.

Example 2. Evaluate

$$\int_0^2 \int_0^x (x^2y - xy^2)\,dydx$$

by all three methods, using $n = 20$, $m = 20$ for the trapezoidal and Simpson's rules, and $\varepsilon = .00001$ for the Romberg integration.

Solution. From the program menu press |T| for the trapezoidal rule, then |C| to enter change mode. Enter

F(X,Y) = X * Y * (X - Y)
G(X) = 0
H(X) = X
A = 0 B = 2
N = 20 M = 20

Then press |G| for the integration.

The values for the inner integrals appear more slowly than they did in Example 1 because M and N are larger. The final result is $T_{20,20}$ = 1.06843223.

As in Example 1, the exact value of the integral can be found by direct integration to be 16/15 = 1.066666. Thus, the trapezoidal rule yields only 2-place accuracy in this example.

Press |RETURN| to return to the program menu. Call up Simpson's rule, press |C|, and change N and M to 20. Then press |G| for the integration. The final result is $S_{20,20}$ = 1.06667111, which is accurate to 5 places.

Now return to the main menu, and press |R| and then |G| for the Romberg integration. The final result, R = 1.06666667, is reached at R2.

In this example the Romberg integration appears to be the best choice. Simpson's rule does not return the exact value of the integral since a term involving x^4 appears in the second integration. However, the Simpson result can be improved substantially. Since the integrand for the inner integration is quadratic in y, Simpson's rule is exact for each integral, even with m = 2, so accuracy can be improved with little cost of computing time. For example, with n = 120 and m = 2, the number of evaluations of f(x,y) required is 121 x 3 = 363, which is less than the 21^2 = 441 required with n = m = 20. Try it: Return to the Simpson's rule menu, enter the new values of N and M, and carry out the integration. The values of the inner integrals appear quickly, and the final result is $S_{120,2}$ = 1.06666666.

Example 3. Evaluate

$$\int_0^{\pi/2}\int_0^{\pi/2-x}(1 - \sin(x + y))dydx$$

using $n = 20$, $m = 20$ for the trapezoidal and Simpson's rules, and $\varepsilon = 10^{-8}$ for the Romberg integration.

Solution. From the program menu press |T| and enter

```
F(X,Y) = 1 - SIN(X + Y)
G(X) = 0
H(X) = PI/2 - X
   A = 0        B = PI/2
   N = 20       M = 20
```

Press |G| for the integration. The final result is $T_{20,20} = .234452829$.

When the integral is evaluated directly its value is found to be $\pi^2/8 - 1$, which is .23370055 to 8 places. Thus, the trapezoidal is accurate to 3 places.

Simpson's rule yields $S_{20,20} = .23700276$, accurate to 5 places, and the Romberg approximation is R = .233702347, also accurate to 5 places.

Example 4. Evaluate

$$\int_0^{1}\int_0^{\arctan\theta}\sin r\theta\, drd\theta$$

using $n = 20$, $m = 20$ for the trapezoidal and Simpson's rules, and $\varepsilon = .00001$ for the Romberg integration.

Solution. Since changing the names of the variables of integration does not affect the value of the integral, the integral is also given by

$$\int_0^{1}\int_0^{\arctan x}\sin xy\, dydx.$$

From the program menu press |T| and enter

```
F(X,Y) = SIN(X * Y)
G(X) = 0
H(X) = ATN(X)
   A = 0        B = 1
   N = 20       M = 20
```

Then press |G| for the integration. The final result for the trapezoidal rule is $T_{20,20}$ = .0868238427.

In this case a good approximation to the exact value of the integral can be found by first carrying out the inner integration to obtain

$$\int_0^{\arctan x} \sin xy\, dy = [-(\cos xy)/x]_0^{\arctan x}$$
$$= (1 - \cos(x \arctan x))/x,$$

then using the program described in Chapter M to obtain

$$\int_0^1 (1 - \cos(x \arctan x)/x\, dx = .086701167,$$

accurate to 9 places. Thus we can see that the trapezoidal rule yields 3-place accuracy in this example.

Now complete the example by carrying out the Simpson and Romberg integrations. The values returned for the integral should be $S_{20,20}$ = .0867010025, and R = .0867011618, the latter obtained as R_4.

4. CONCLUSIONS

The examples above tend to confirm our previous experience that Simpson's rule appears to be more accurate than the trapezoidal rule for many elementary functions. The reason is that trapezoidal errors are of the order of the square of the step size, as compared with the fourth power for Simpson's rule. The Romberg method errors are also of the order of the fourth power of the step size, but some accuracy is lost in the program's implementation in the trade-off for speed.

As we saw in Example 2, it may be possible to improve the results of the trapezoidal rule and Simpson's rule by a judicious choice of the number of subintervals. In this connection, success requires a little thought about the nature of the integrand, the integrations involved, and the properties of the numerical methods.

Checking the value found for a double integral against an "answer book" usually shows only whether an error has been made in the calculation, giving little idea as to the location of any error. When an error is detected this way, the only alternatives are often to recheck all calculations or to redo the entire problem. DOUBLE INTEGRAL enables you to examine the values of the y-integrals I_i. A spot check of these results for one or two values of i may give a quick indication of where the error occurred, and may sometimes even suggest the correct result.

PROBLEMS

1. Approximate the integral in Example 3 with $n = m = 40$. Simpson's rule now gives better results than R, primarily because of the programming choice to limit the Romberg computation to seven extrapolation rows. (To ten places, $\pi^2/8 - 1 = .2337005501$.)

2. Evaluate the integral of Example 4 by Simpson's rule (a) with $n = m = 30$, (b) with $n = m = 40$. To how many decimal places is the answer accurate in each case? (To nine places, the value of the integral is .086701167.)

Approximate the integrals in Problems 3-16. Try all three numerical methods. Experiment with n, m, and ε to develop a feeling for the accuracy of each approximation.

3. $\int_0^1 \int_0^1 (x + y)\,dy\,dx$; try $n = m = 1$ for the trapezoidal rule, $n = m = 2$ for Simpson's rule.

4. $\int_0^1 \int_x^1 (x + y)\,dydx$; try n = 50, m = 1 for the trapezoidal rule.

5. $\int_0^1 \int_0^x (2x + y)\,dydx$; try n = 2, m = 2 for Simpson's rule.

6. $\int_0^4 \int_0^{2-x/2} (4 - x - 27)\,dydx$

7. $\int_1^2 \int_0^x xy\,dydx$

8. $\int_0^1 \int_0^x (xy + y^2)\,dydx$

9. $\int_1^3 \int_1^x \frac{1}{xy}\,dydx$

10. $\int_0^{\pi/4} \int_0^{4\cos 2\theta} r\,drd\theta$

11. $\int_0^2 \int_0^{y+1} (x + y)\,dxdy$

12. $\int_0^1 \int_0^x e^{-(x+y)}\,dydx$

13. $\int_0^1 \int_0^{x/2} \cosh(x - y)\,dydx$

14. $\int_0^{\pi/2} \int_1^{2+\cos\theta} (\theta + \ln r)\,rdrd\theta$

15. $\int_0^1 \int_0^1 \arctan xy\,dydx$

16. $\int_0^1 \int_0^{x^2} (x^2 + y^2)\,dydx$

V. Scalar Fields

1. PURPOSE

This program enables you to study analytical and physical properties of scalar fields defined on square regions in the XY-plane.

2. DESCRIPTION

A scalar field is a function F that assigns a real number, or scalar, F(X,Y) to every point (X,Y) in its domain. Such a function may represent a temperature distribution, an electric field potential, or the pressure within a region through which a compressible fluid is flowing.

After entering the function and domain you may (1) plot the field with small squares centered at lattice points in the domain showing the magnitude of the field by their size and the sign of the field by their shading, (2) plot grad F, (3) plot div F, (4) calculate the values of F, grad F, and div(grad F) at selected points, and evaluate the line integral $\int$grad(F) . dR along straight-line paths and selected city-block paths in the domain of F.

3. STEP BY STEP

Load the program, read the greeting messages, and go on to the program menu shown in Screen 1. After reading the

menu, press |F| to call up the field plot menu shown in Screen 2.

```
|F|IELD:  F(X,Y)
|G|RADIENT:   GRAD F *
|L|APLACIAN:  DIV(GRAD F)
|V|ALUES OF F, G, L
|D|IRECT PATH LINE INTEGRAL OF GRAD F . DR
|Q|UIT
|_|

* INCLUDES LINE INTEGRAL OF GRAD F . DR
OVER PATH TRACED AS YOU STEER CURSOR
```

Screen 1. The program menu.

```
SCALAR FIELD
F(X,Y) = X * Y

GRAPH WINDOW CENTER
X0 = 0                 Y0 = 0

WINDOW WIDTH       LATTICE SPACING (10-30)
 W = 10                 H = 20

ESTIMATED POINT IN WINDOW
AT WHICH MAX. ABS(F(X,Y)) OCCURS
 X = 0                  Y = 0

|C|HANGE ENTRY   |G|O ON   |M|ENU   |Q|UIT
```

Screen 2. The field plot menu.

The values displayed in Screen 2 show that the current plot window is a square whose center is (X0,Y0) = (0,0) and whose sides are 10 units long.

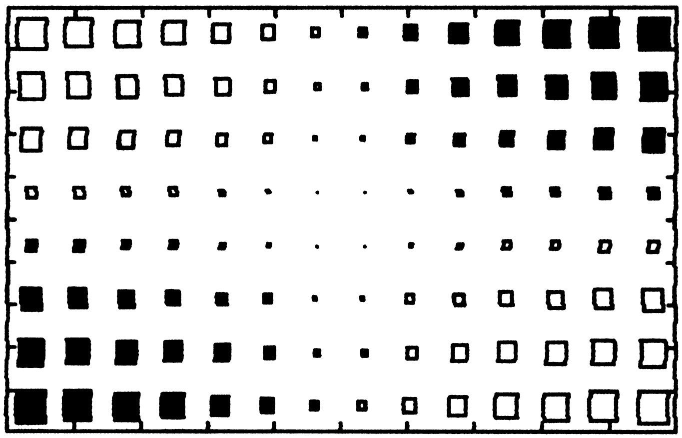
The lattice point spacing is the number of screen pixels that will separate the centers of contiguous squares in the plot. The value H = 10 gives the finest resolution (closest packing), while H = 30 spreads the squares farthest apart. There is a trade-off between speed and resolution. With H = 30 you get a rough picture fast. With H = 10 you get a finer picture, but you may have to wait. When in doubt, start with H = 15 or H = 20.

The estimated location of the maximum value of |F| tells the computer in advance how to assign square sizes to function magnitudes in the plot. The largest possible square size will be used to show function magnitudes close to the magnitude of F at this point. Larger magnitudes will also be shown with this square size; smaller magnitudes will be shown with smaller square sizes. The location of a point where |F| is maximal will always be shown initially as the graph window center, and you may wish to enter a new location before the plotting begins. However, do not worry if you are unsure of where |F| is largest. You can always ask the computer to rescale the plot properly after it has calculated the function values for the first display.

Example 1. The scalar field F(X,Y) = X * Y

This is the program's default example, and, except for the location of a point where |F| is maximal, the field plot menu shown in Screen 2 is ready to go.

Press |C| followed by five |RETURN|s and press |5| |RETURN| |5| |RETURN| to enter X = 5 and Y = 5 as a point where |F| is maximal. In the forthcoming plot, the largest square size will be used to show magnitudes greater than or equal to |F(5,5)| = 5*5 = 25. Press |G| to plot the field (Screen 3).



CENTER (0,0) UNIT = 1

MAX. ABSOLUTE VALUE OF FIELD APPROX. AT

X,Y = -4.64,4.38

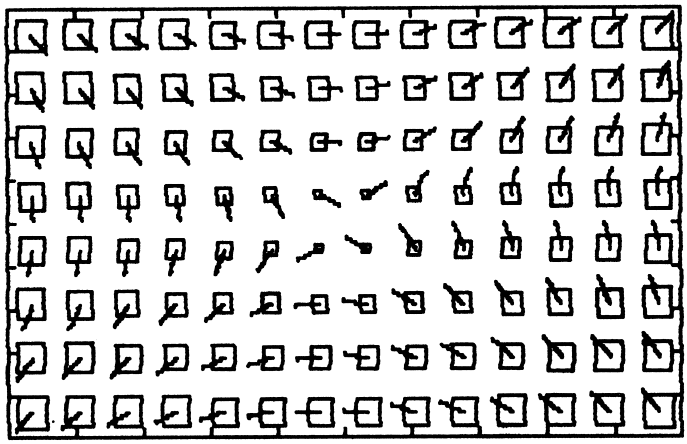
|R|ESCALE |M|ENU |Q|UIT |_|

Screen 3. The field plot for F(X,Y) = X*Y, with window center (0,0), window width W = 10, and square spacing H = 20 screen pixels.

The solid squares in Screen 3 show positive values of F, the hollow squares negative values. The ticks around the window frame are one unit apart (screen width W = 10, divided by 10).

To view the gradient field, press |M| to return to the main menu. Then press |G| for the gradient plot menu. This menu is the field plot menu (Screen 2) without the request for a max |F| location.

Press |G| to plot the field. When you are ready, press |P| to add the vector magnitudes to the plot. They will be shown as small squares around the initial points of the vector elements, as shown in Screen 4.



CENTER (0,0) UNIT = 1

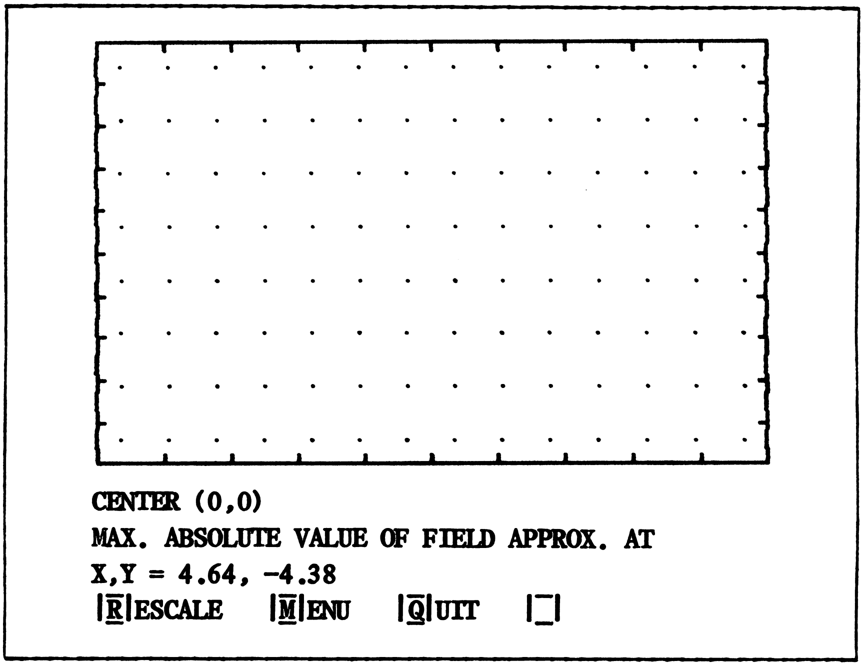
|P|LOT MAGNITUDE |E|RASE MAGNITUDE

|L|INE INTEGRAL |M|ENU |Q|UIT |_|

Screen 4. This plot of the gradient field of F(X,Y) = X*Y shows both magnitude and direction.

Press |E| to erase the magnitude squares. This feature is included for the convenience of those who wish to see the magnitudes first but do not want them during the cursor-steered line integrations, which use this graphics screen.

Now begin the arrangements for plotting the divergence of grad F by pressing |M| and |L|. When the Laplacian menu appears (identical with field plot menu in Screen 2 except for the replacement of MAX. ABS(F(X,Y)) by MAX. ABS(DIV(GRAD F)) in the third line from the bottom), press |G| to plot the Laplacian with the current values. Since the Laplacian is identically zero in this example, there is no point in estimating the location of the field maximum. The plot (Screen 5) will consist of small square dots of uniform size.



Screen 5. The plot of div(grad(X*Y)) = div(YI + XJ) = 0.

After viewing the divergence plot, press |M| to prepare for the next example.

Example 2. The electric dipole field

F(X,Y) = X/(X*X + Y*Y)^1.5

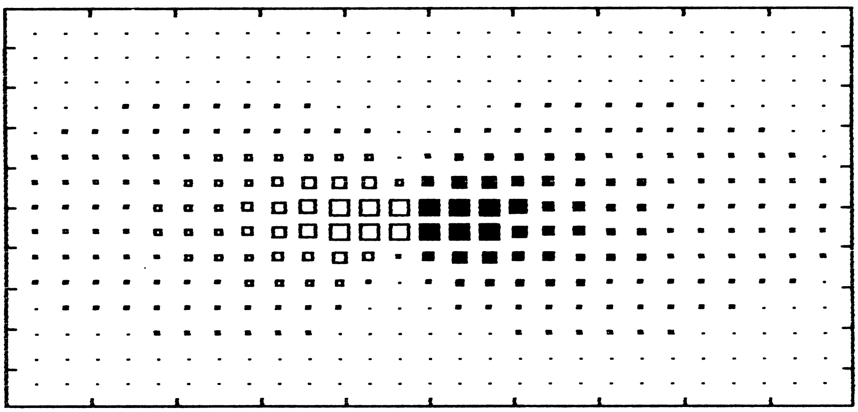
Press |F| from the main menu, then |C|, and enter in the formula for F. When you are satisfied with your entry, press |RETURN| and enter

X0 = .1 Y0 = .1 W = 4 H = 10

one at a time, followed by |RETURN|s. The field is not defined at (0,0), so you must center the field somewhere else. The point (.1,.1) is far enough away to avoid arithmetic trouble, but close enough to preserve important

symmetries in the plot. Note that the default estimate of the location of max|F| changes to (.1,.1) as X0 and Y0 are changed.

Enter X = .3, Y = .3 as the estimate of the location of MAX|F|. Then press |G| to plot the field. The result should look like the plot in Screen 6.



CENTER (.1,.1) UNIT = .4
MAX. ABSOLUTE VALUE OF FIELD APPROX. AT
X,Y = .1,.1
|R|ESCALE |M|ENU |Q|UIT |_|

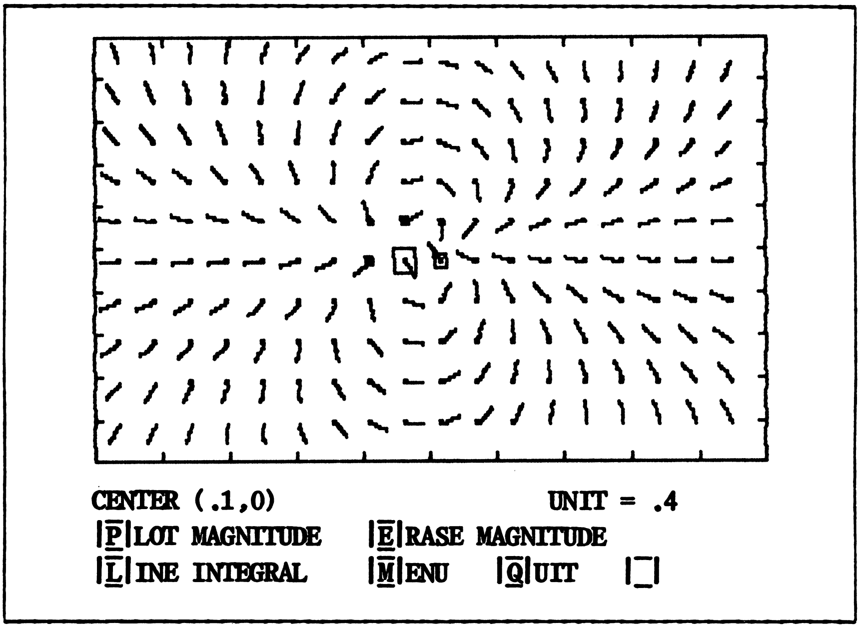
Screen 6. A plot of the dipole field F(X,Y) = X/(X*X + Y*Y)^1.5. The "cosine" lobes are clearly shown.

Now press |M|, |G|, and then |C|, and enter

X0 = .1 Y0 = 0 W = 4 H = 15

for the gradient plot. After checking your values, press |G| to display the gradient field. When the display is complete, press |P| to include the squares that show the vector magnitudes. No action will be visible for some time

because the magnitudes away from the center are so small that the magnitude plots are covered by the direction plots. The final result should look like the display in Screen 7.



Screen 7. The gradient of the dipole field in Screen 6.

The field lines in the gradient display diverge from the positive charge and reconverge at the negative charge.

To calculate line integrals in the gradient field, press |E| to erase the magnitudes, then |L| for the line integral. Enter X = 1, Y = 1 for the starting point. A cross (+) will appear at the point (1,1) and the message at the bottom of the screen will change to

LINE INTEGRAL TO 1,1 = 0
|U|P |D|OWN |L|EFT |R|IGHT |N|EW PATH |M|ENU |_|

Press enough |L|s and |D|s to steer the cursor to the point (-1,0). The motion will be slow because of the time required by the function evaluations. The computer will

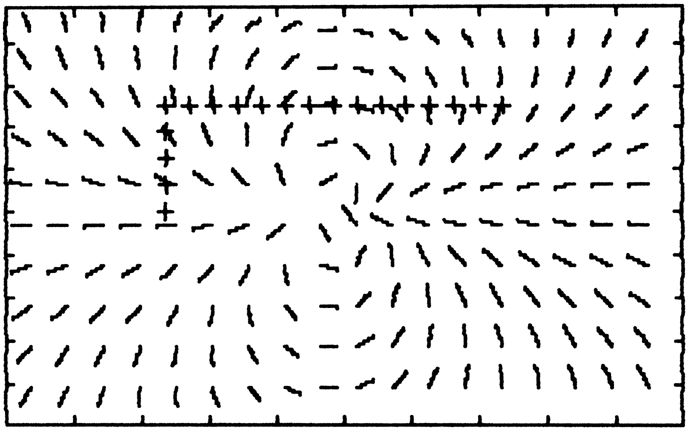
return the result -1.37 for the integral of GRAD F . DR along the path traced by the cursor (Screen 8).

Now press |M| for the program menu and |D| to calculate a direct (straight-line) path integral. Enter

X1 = 1   Y1 = 1   X2 = -1   Y2 = 0

and press |G| to start the computation. After about 45 seconds, the computer will return the result -1.37.

This concludes the demonstration.



START PATH AT X = 1 Y = 1
UNIT = .4
LINE INTEGRAL TO -1,0 = -1.37
|U|P |D|OWN |L|EFT |R|IGHT |N|EW PATH |M|ENU

Screen 8. The value of the line integral along the marked path from (1,1) to (-1,0) is -1.37.

4. |ESC|

To stop a plot in progress, press |ESC|. The option lines will reappear at the bottom of the screen, and you may continue as you normally would from there.

To plot the gradient line elements and magnitudes simultaneously, press |ESC| after the gradient line element plot has begun and the computer has run long enough to get some idea about how the magnitudes are running. Then press |P|. The plotting will begin afresh, with line elements and magnitude squares together.

PROBLEMS

1. Plot the Laplacian (divergence of the gradient) of F(X,Y) = X/(X*X + Y*Y) 1.5, with X0 = .1, Y0 = 0, W = 4, H = 10. Then press |R| to rescale the plot.

2. a) To investigate the effect of changing the estimate of the location of max|F| on field element scaling, plot the field F(X,Y) = 1/(X*X + Y*Y) with center (.1,0), W = 2, H = 15, and max locations (.1,0), (.2,.2), and (.01,.01).
   b) Plot div(grad(F)) with H = 20.

Carry out the steps of Examples 1 and 2 with the functions and parameter settings in Problems 3-15. Rescale the divergence plots, if necessary. Experiment with different parameter settings and integration paths.

3. F(X) = |X|, center (.1,.1), W = 2, H = 20

4. F(X) = |X|+|Y|, center (.1,.1), W = 10, H = 10

5. F(X) = |X|-|Y|, center (.1,.1), W = 10, H = 10

6. F(X) = SIN(X), center (0,0), W = 20, H = 15

7. F(X) = SIN(X+Y), center (0,0), W = 20, H = 15, max location estimate (.7,.7)

8. F(X,Y) = 1/(X*X + Y*Y) .5, center (.1,.1), W = 5, H = 10, max location estimate (.1,0)

9. F(X,Y) = Y/(X*X + 1), center (0,0), W = 8, H = 10, max location estimate (0,4). For the gradient plot, also try W = 4.

10. F(X,Y) = (X+Y)/(X*X+1), center (0,0),
W = 8, H = 10, max location estimate (0,5).
Also try W = 4.

11. F(X,Y) = (X+Y)/(2+COS(X)), center (0,0), W = 20
H = 10, max location estimate (-16,-18).

12. F(X,Y) = X*Y/(X*X+Y*Y), center (.1,.1), W = 10
H = 15, max location estimate (1,1).

13. F(X,Y) = X + X/(X*X+Y*Y) 1.5, center (.1,.1),
W = 8, H = 10, max location estimate (.2,0)

14. F(X,Y) = EXP(-X/5)*COS(Y - X), center (5,5),
W = 10, H = 10, max location estimate (0,0).

15. F(X,Y) = X*X*Y/(X 4 + Y 2), center (.1,.1),
W = 10, H = 10, max location estimate (-1.33,-1.77)
(Avoid the line integral; it's very slow.)

W. Vector Fields

1. PURPOSE

This program enables you to study analytical and physical properties of vector fields of the form M(X,Y)*I + N(X,Y)*J defined on square regions in the XY-plane.

2. DESCRIPTION

After entering the component functions M(X,Y) and N(X,Y), you may plot the field over any chosen rectangular region, plot the field's curl and divergence, and evaluate the field, curl, and divergence at requested points. You may also calculate the surface integral of the curl over a given rectangular region and the flux of the field across the boundary of a coordinate rectangle. Finally, you may calculate the line integral of the field along a directed line segment joining two points or along a variety of city-block paths between two points in the field's domain.

3. STEP BY STEP

Example 1. The vector field F(X,Y) = Y * I + X * J and its flux integral.

This is the program's default example. To start, load the program from the disk menu, read the greeting messages, and continue on to the program menu, shown in Screen 1.

After reading the program menu, press |F| to call up the field menu, shown in Screen 2.

```
|F|IELD:  F(X,Y) *
|C|URL F
|D|IV F
|V|ALUES OF F, C, D
|S|URFACE INTEGRAL OF CURL F . N
|L|INE INTEGRAL OF F . N (FLUX)
|A|RROW PATH LINE INTEGRAL OF F . DR
|Q|UIT
|_|

* INCLUDES LINE INTEGRAL OF F . DR
  OVER YOUR CURSOR-STEERED PATH
```

Screen 1 The program menu.

```
FIELD F(X,Y) = M(X,Y)*I + N(X,Y)*J
M(X,Y) = Y
N(X,Y) = X
GRAPH CENTER, WIDTH, SPACING (10-30)
XO = 0                 YO = 0
 W = 10                 H = 20
|C|HANGE ENTRY   |G|O ON   |M|ENU   |Q|UIT   |_|
```

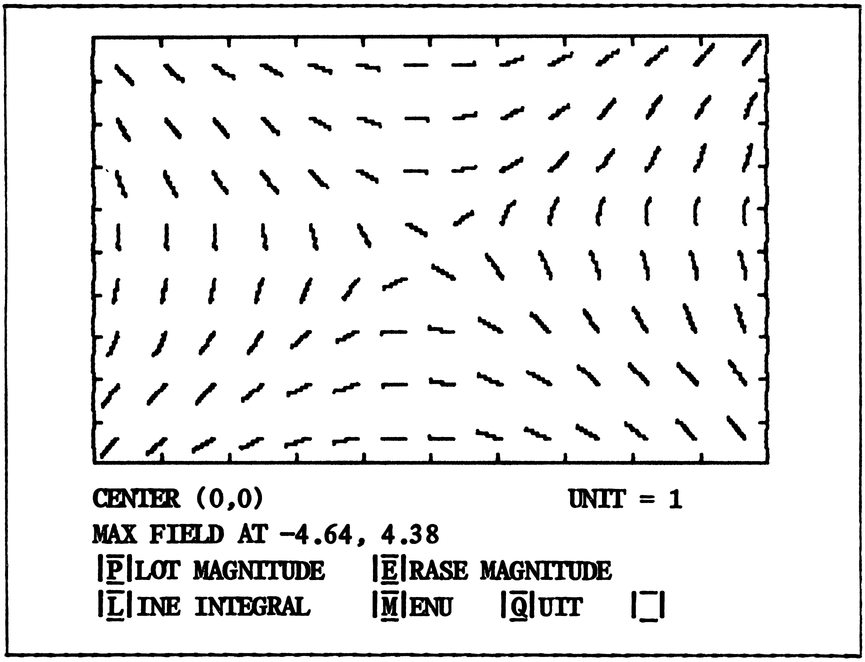
Screen 2. The field menu.

The values in Screen 2 show that the current plot window is a square whose center is (XO,YO) = (0,0) and whose sides are W = 10 units long.

The spacing H is the number of screen pixels that will separate the initial points of the plotted vectors. The

value H = 10 gives the finest resolution (closest packing), while H = 30 spaces the vector's initial points farthest apart. With H = 30 you get a rough picture fast. With H = 10 you get a finer picture, but you may have to wait. When in doubt, start with H = 15 or H = 20.

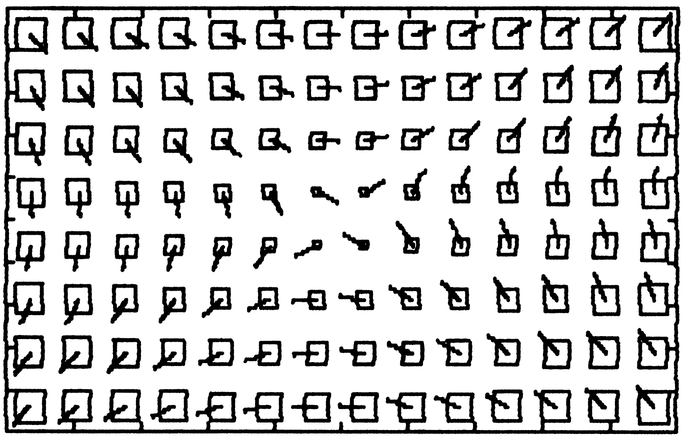
To continue the demonstration, press |G| to accept the current values and plot the field. The plot will look like the one in Screen 3.



Screen 3. Vectors from the field Y * I + X * J in the square window with center (0,0) and width W = 10. The initial points are H = 15 screen pixels apart.

The data at the bottom of the display in Screen 3 tell you that one of the largest vectors plotted has its initial point at (-4.64, 4.38). The ticks around the window frame are one unit apart (screen width 10, divided by 10).

Press |P| to add the vector magnitudes to the plot. They will be shown as squares of different sizes centered at the initial points of the vectors, as in Screen 4.



CENTER (0,0) UNIT = 1
MAX FIELD AT -4.64, 4.38
|P|LOT MAGNITUDE |E|RASE MAGNITUDE
|L|INE INTEGRAL |M|ENU |Q|UIT |_|

Screen 4. This plot of the field Y * I + X * J shows both magnitude and direction.

Now press |M| to return to the program menu, and notice the additional option

|P|REVIOUS GRAPH.

Press |P| to return to Screen 4, and press |RETURN| to recall the program menu.

Press |V| to calculate some of the vector and numerical values of the field. When the data entry menu appears

(upper portion of Screen 5), press |C| to change entries. You now have an opportunity to change the field, but, to continue the demonstration, press two |RETURN|s to keep the current component functions. Then press |3| |RETURN| |1| |RETURN| to enter the coordinates (3,1) of the domain point at which the evaluations will take place. The values computed will appear in the lower portion of the display, as shown now in Screen 5.

```
FIELD F(X,Y) = M(X,Y)*I + N(X,Y)*J
M(X,Y) = Y
N(X,Y) = X
DOMAIN POINT
 X = |3|              Y = 1

F(X,Y) = 1I + 3J
MAGNITUDE = 3.16
DIRECTION = .32I + .95J
CURL F = 0 K
DIV = 0

|RETURN| ACCEPT ENTRY    |ESC| ABORT ENTRY
```

Screen 5. Function values at the point (3,1).

The cursor is blinking at the X-value of the domain point, and you may enter additional values for X and Y as you wish. Try some. When you are ready to go on, press |ESC| to leave entry mode and |M| to return to the program menu.

Press |L| on the program menu to try the line integral for the flux of the field. The display will change to the one shown in Screen 6.

```
FIELD F(X,Y) = M(X,Y)*I + N(X,Y)*J
M(X,Y) = Y
N(X,Y) = X
CENTER OF RECTANGULAR REGION
 X = 0                Y = 0
HORIZONTAL, VERTICAL DISTANCE TO SIDES
 H = 5                V = 5

|C|HANGE ENTRY   |G|O ON   |M|ENU   |Q|UIT   |_|
```

Screen 6. The flux integral screen.

The flux here is $\int F \cdot n\,ds$, calculated around a rectangle whose sides are parallel to the coordinate axes in the XY-plane. The motion is counterclockwise, and n is the outer unit normal.

The default rectangle is a ten by ten square centered at the origin. To calculate the outward flux of F across it, press |G|. After a brief pause the computer will respond with the message

FLUX INTEGRAL OF F = 0

Now enter

X = 2 Y = 2 H = 5 V = 6

and press |G|. The outward flux across this rectangle is 88 units.

Key in other values as you please and, when ready, press |ESC| |M| to leave entry mode and prepare for the next example.

Example 2. The curl of the vector field
F = -Y * Y * I + X * X * J.

Press |C| on the program menu to call up the curl menu. Then press |C| to change values and enter the components of

F by pressing

|=| |Y| |*| |Y| |RETURN| and |X| |*| |X| |RETURN|

At this point, the curl menu will look like the one in Screen 7 except perhaps for the coordinates of the graph center (which will be the ones you were using last) and the estimated point in the graphing window at which the maximum magnitude of CURL(F) occurs, whose default coordinates will always be those of the initial graph center.

```
FIELD F(X,Y) = M(X,Y)*I + N(X,Y)*J
M(X,Y) = -Y*Y
N(X,Y) = X*X

GRAPH CENTER, WIDTH, SPACING (10-30)
 X = |2|               Y0 = 2
 W = 10                 H = 20

ESTIMATED POINT IN WINDOW
AT WHICH MAX MAG(CURL F) OCCURS
 X = 2                  Y = 2

|C|HANGE ENTRY   |G|O ON   |M|ENU   |Q|UIT   |_|
```

Screen 7. The curl menu.

The curl menu is the field menu with the added request for an estimate of the location of the maximum field magnitude in the plotting window. The coordinates of this location tell the computer in advance how to assign square sizes in the plot of curl magnitudes, which, except for sign, is all you will see of the curl vector since it is perpendicular to the XY-plane. Positive K components are indicated by solid squares, negative K components by hollow squares.

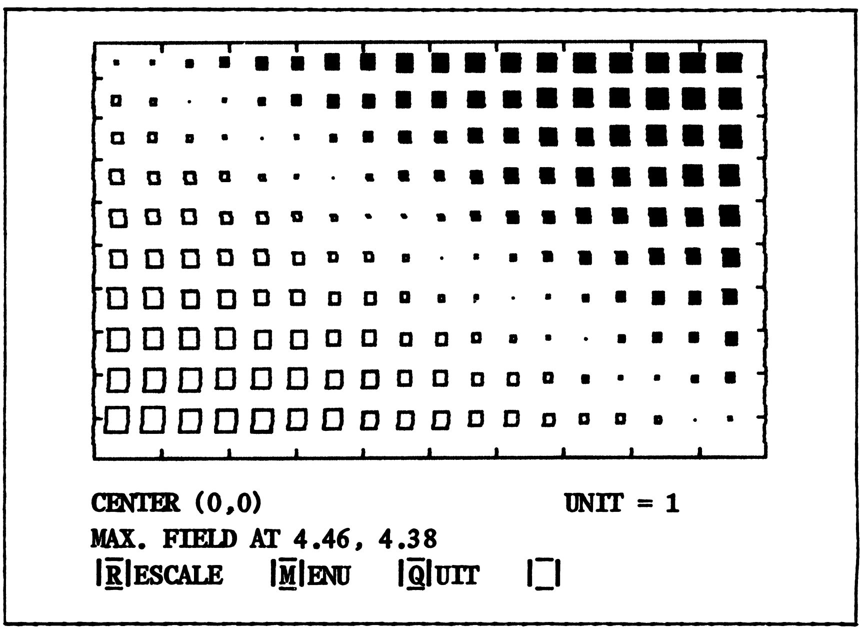
The coordinates of the estimated max location will always coincide with those of the center when the plot curl menu first appears. This point is sure to be in all plotting windows. Do not worry if you do not know what to

enter instead. Just enter the coordinates of some point in the domain of the field where the curl is not zero. As you are about to see, you can have the computer rescale the plot after it has calculated the curl magnitude values for the first display. In this example, the K-component of curl F is 2X + 2Y, and we can enter (1,1) as a point where curl F ≠ 0K (zero times K).

To continue the example, enter

X0 = 0   Y0 = 0   W = 10   H = 15

and then press |1| |RETURN| |1| |RETURN| to enter the coordinates of the point (1,1) as a point where curl F ≠ 0. Then press |G| to plot the curl magnitudes. The cursor will reappear when the plot is complete. When it does, press |R| to rescale the plot. The rescaled plot should look like the one in Screen 8.



Screen 8. The rescaled magnitude plot of curl (-Y*Y*I + X*X*J); center (0,0), W = 10, H = 15.

Now press |M| to return to the main menu, and press |D| to call up the menu for plotting the divergence of F. This menu is identical with the plot curl menu shown in Screen 7 except for the use of ABS(DIV F) in place of MAG(CURL F) in the estimated max location prompt. In responding to this prompt you would normally enter the coordinates of a point where the divergence is not zero. In the present example, however, DIV(-Y*Y*I + X*X*J) is zero at every point and you may as well accept the default location estimate X = 0, Y = 0.

Press |G| to plot the field. The plot should appear as a rectangular array of dots of uniform size. When the display is complete press |M| to return to the main menu.

Starting from the main menu, press |A| to display the straight-line path integral menu, shown here as Screen 9.

```
FIELD F(X,Y) = M(X,Y)*I + N(X,Y)*J
M(X,Y) = -Y*Y
N(X,Y) = X*X
INITIAL POINT
X1 = 0              Y1 = 0
TERMINAL POINT
X2 = 0              Y2 = 0


|C|HANGE ENTRY   |G|O ON   |M|ENU   |Q|UIT   |_|
```

Screen 9. The straight-line path integral menu.

The menu shown in Screen 9 enables you to calculate the integral of F . DR along the line segment from the initial point (X1,Y1) to the terminal point (X2,Y2).

To calculate

$$\int_{(0,0)}^{(2,3)} (-Y*Y*I + X*X*J) \,.\, DR$$

press |C|, four |RETURN|s, |2| |RETURN| |3| |RETURN| |G|. The computer will return the value -1.98.

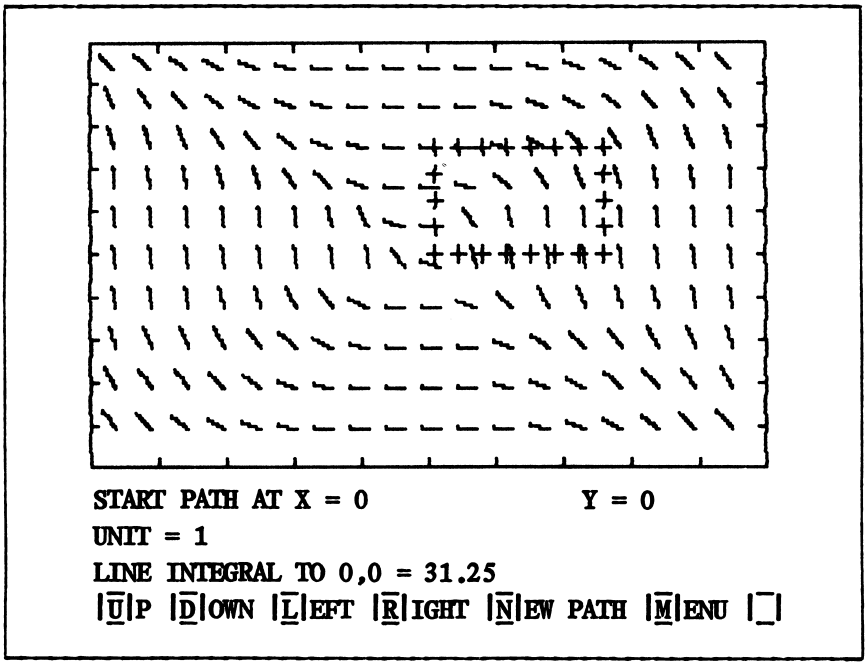
Now press |ESC| to leave entry mode, |M| to return to the program menu, and |F| |G| to plot the field in preparation for calculating the integral of the field over the path traced by the cursor as you steer it around the plotting window. When the plot is complete, press |L| and enter 0 for both coordinates of the initial point of the path. A cross (+) will appear at the point (0,0) in the center of the field, and the lines

LINE INTEGRAL TO 0,0 = 0
|U|P |D|OWN |L|EFT |R|IGHT |N|EW PATH |M|ENU

will appear at the bottom of the screen. The keys |U|, |D|, |L|, and |R| steer the cross in the indicated directions. Pressing |N| enables you to enter a new starting point, and pressing |M| recalls the program menu.

Press |R| enough times to steer the cursor to the point (2.5,0) and press |U| enough times to reach (2.5,2.5). Watch the integral's values change (bottom line of screen) as the cursor moves. The value of the line integral from (0,0) to (2.5,2.5) along the route taken should be given as 15.62.

Continue the line integration by pressing |L| to move the cursor left to the point (0,2.5). Then return the cursor to the origin by pressing |D|. At this point the display should look like the one in Screen 10. The value of the line integral around the closed path that the cursor has just traversed is 31.25. The field -Y*Y*I + X*X*J is not a conservative field.



Screen 10. This plot of F = -Y*Y*I + X*X*J leaves out the vector magnitude squares shown in Screen 8. The integration path, marked by crosses (+), is easier to see that way. The value of the line integral of F counterclockwise around the indicated rectangle is 31.25.

4. |ESC|

To stop a plot in progress, press |ESC|. The option lines will reappear at the bottom of the screen and you may continue as you normally would from there.

To plot vector line elements and magnitudes simultaneously, press |ESC| after the vector field plot has run a short while. When the option lines reappear, press |P|. The plotting will begin afresh, with line elements and magnitudes together.

PROBLEMS

1. Continuation of Example 2.

   F = -Y*Y*I + X*X*J
   Plot window: Center (0,0), W = 10, H = 15

   a) Calculate the line integral of F <u>clockwise</u> around the rectangular path shown in Screen 10, starting at (0,0).
   b) Calculate the line integral of F counterclockwise around the rectangular path shown in Screen 10, starting at the point (2.5,0) instead of (0,0). Is the integral's value still 31.25?
   c) Calculate the surface integral of curl F over the 4 unit by 4 unit rectangles centered at (1,1) and (0,0).
   d) Calculate the flux integral of F counterclockwise around the 4 unit by 4 unit rectangles centered at (1,1) and (0,0). What values would you expect the <u>clockwise</u> flux integrals to have?

In Problems 2–4, carry out the steps of Examples 1 and 2 for the field over the given plot window. Experiment with additional field values and integration paths. Rescale magnitude plots where necessary.

2. F = I + J
   Plot window: Center (5,5), W = 10, H = 10

3. F = Y*I - X*J
   Plot window: Center (0,0), W = 10, H = 15

4. F = -(X/(X^2 + Y^2))*I - (Y/(X^2 + Y^2))*J
   Plot window: Center (.1,.1), W = 10, H = 20

5. F = (Y/(X^2 + Y^2)*I - (X/(X^2 + Y^2)*J
   Plot window: Center (.1,.1), W = 10, H = 20

X. First Order Initial Value Problem

1. DESCRIPTION

This program provides numerical solutions to problems of the form

$$y' = f(x,y), \quad y(a) = y_0$$

over an interval $a \leq x \leq b$ with a fourth order Runge-Kutta method with up to one hundred steps. You enter f, a, b, y(a), and the number of steps. After executing the solution routine you may elect to see graphs and value tables of both y and y'. These tables may also be printed.

2. FIRST ORDER INITIAL VALUE PROBLEMS

First order initial value problems usually appear with the definition of the indefinite integral in calculus. In the simplest case, no reference is made to an interval, the formula for y' depends on x alone, and the problem takes the form

$$y' = f(x); \quad y(a) = y_0.$$

Initial efforts are often focused on finding a closed form solution to the problem: we find a formula for y,

$$y = \int f(x)dx + C$$

and then substitute a for x and y_0 for y to find C.

For many initial value problems there are satisfactory techniques for finding closed form solutions. However, many problems are difficult or even impossible to solve in closed form, and such problems are often solved numerically, on a computer, over a finite interval $a \leq x \leq b$.

The simplest numerical methods for solving first order initial value problems involve computing a table of values $w_0, w_1, \ldots, w_n$ corresponding to points $x_0, x_1, \ldots, x_n$ in $[a,b]$. The values w_i approximate the corresponding values of the true solution, $y = y(x)$, of the given problem, i.e.,

$$w_i \quad y(x_i) \quad \text{for} \quad i = 0, 1, \ldots, n.$$

The present Toolkit program uses a fourth order Runge-Kutta method to obtain the w_i. Under this method a value of n is chosen, and the points $x_0, x_1, \ldots, x_n$ are equally spaced across the interval. The distance between consecutive points is called the step size and is denoted by h:

$$h = (b - a)/n,$$

and the values x_i are

$$x_0 = a,\ x_1 = a + h,\ x_2 = a + 2h,\ \ldots,\ x_n = a + nh = b.$$

The values of w_i are found as follows: let

$$w_0 = y_0,$$

and for $i = 0, 1, \ldots, n - 1$ let

$$w_{i+1} = w_i + (k_1 + 2k_2 + 2k_3 + k_4)/6,$$

where

$$\begin{aligned} k_1 &= hf(x_i, w_i) \\ k_2 &= hf(x_i + h/2, w_i + k_1/2) \\ k_3 &= hf(x_i + h/2, w_i + k_2/2) \\ k_4 &= hf(x_i + h, w_i + k_3). \end{aligned}$$

Thus, the solution procedure begins at $x_0 = a$, with the given initial condition used for the first value: $w_0 = y_0$. Then a horizontal step of h units is taken from x_0 to x_1, and

the corresponding vertical step from w_0 to w_1 is h times a weighted average of four values of y' near the starting point. (See Fig. 1.) The procedure is continued across the interval and terminated at $x_n = b$.

The method is relatively accurate, with an error of order h^5 at each step, and h^4 across the interval. The cost of this accuracy is computing time, since four function evaluations are required at each step. It takes several seconds to execute the solution routine, even when the number of steps is relatively small. (You can find more about the method in a text on numerical methods.)

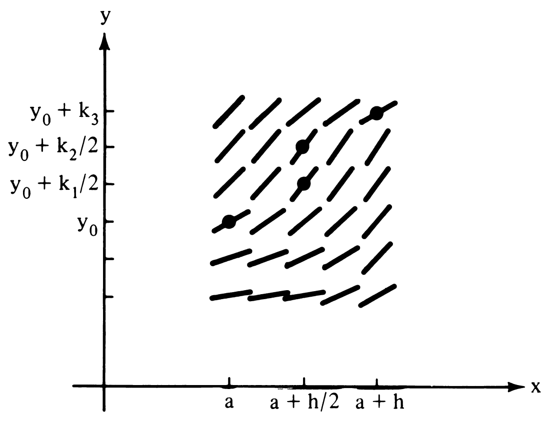


Figure 1. The four points in the first Runge-Kutta step.

3. STEP BY STEP

Load the program from the disk menu, read the greeting messages, and continue on to the problem display shown in Screen 1. This is the setting for our first example.

Example 1. Solve the initial value problem

$$y' = x + y, \quad y(-1) = 1$$

on the interval from -1 to 1, using 20 steps.

```
EQUATION AND ENDPOINTS

    Y' = X + Y

A = -1                   B = 1

INITIAL VALUE        Y(A) = 1
NUMBER OF STEPS         N = 20

|C|HANGE ENTRY    |G|O ON    |Q|UIT
```

Screen 1. The problem display for Example 1.

Solution. The equation and endpoints from this problem are the ones in the initial problem display in Screen 1. When you have reviewed the entries, press |G| to execute the solution routine. In about four seconds, the computer will present the output menu shown in Screen 2.

Press |S| to graph the solution as shown in Screen 3. Screen 3 also shows the values of a and b and the minimum and maximum values computed for y on the interval [a,b]. As x varies from −1 to 1, the computed values vary from 1 to 5.38904477. Since the closed form solution is

$$y = e^{x+1} - x - 1,$$

the difference between the true 8-place value of y(1), 5.38905610, and the computed value is .00001133.

Now press |R| to rescale the graph to the domain and range of the computed solution. Pressing |R| again will reconstruct the original graph (Screen 3).

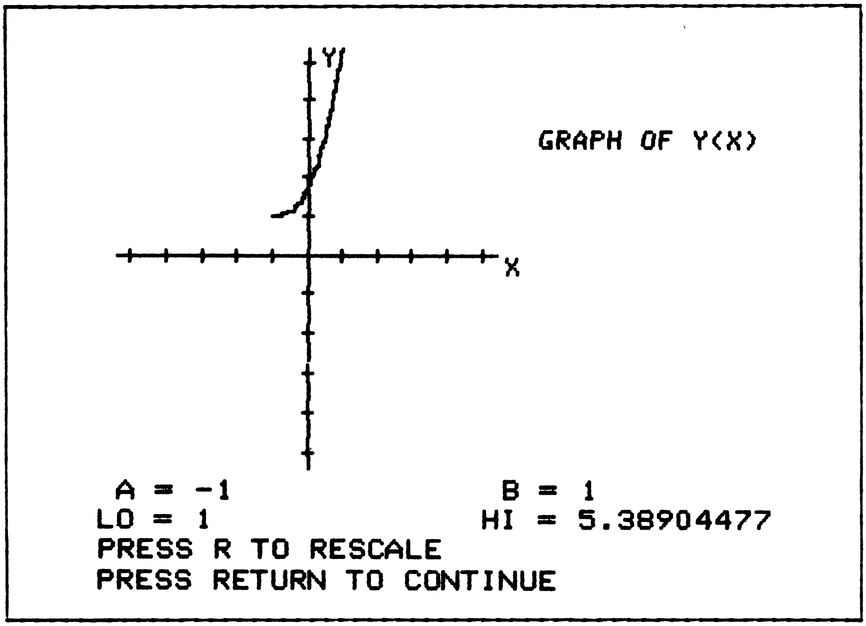
The graph of Y'(X) can be obtained either by pressing |D| directly, or by pressing |RETURN| and then |D| from the output menu. You can "toggle" back and forth between the graphs of Y and Y' by pressing |S| and |D| alternately. The graph of Y' is shown in Screen 4.

OUTPUT MENU

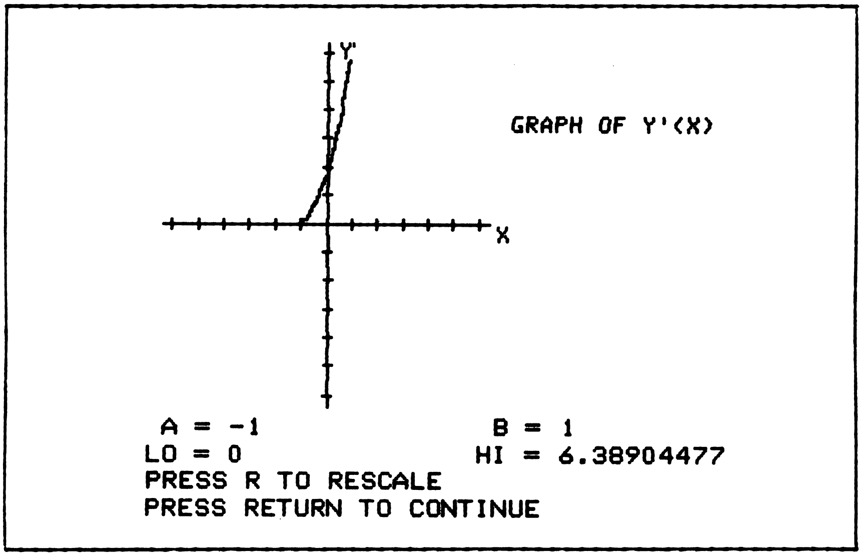
|S| .. SOLUTION: GRAPH OF Y(X)
|D| .. DERIVATIVE: GRAPH OF Y'(X)
|V| .. VALUES FOR Y(X) AND Y'(X)
|H| .. HARD COPY OF VALUES
|P| .. PROBLEM DISPLAY
|Q| .. QUIT

PRESS LETTER OF YOUR CHOICE

Screen 2. The program's output menu.



Screen 3. The graph of the solution Y(X) computed for the problem y' = x + y, y(-1) = 1.



Screen 4. The graph of Y'(X) = EXP(X + 1) - 1.

Complete the demonstration by pressing |RETURN| to return to the program menu and then |V| to display the tables of values of Y and Y'. Pause with a |RETURN| to read any particular segment of the table, and continue with another |RETURN|, as desired. The values displayed for Y are the computed values w_0, w_1, ..., w_{20}. The values displayed for Y' are the numbers

$$f(x_i, w_i) = x_i + w_i.$$

Press |RETURN| to return to the output menu.

If your computer is connected to a printer and you know the slot number of the interface card (the usual number is 1), you may print the values of Y and Y'. To do so, press |H| from the output menu, press the slot number, and press |RETURN| to start printing.

Now return to the output menu, if you are not there already, and press |P| for the problem display to prepare for the next example.

Example 2. Solve the problem

$$y' + y = e^{-x}, \quad y(0) = 0$$

on the interval from 0 to 3, with n = 30.

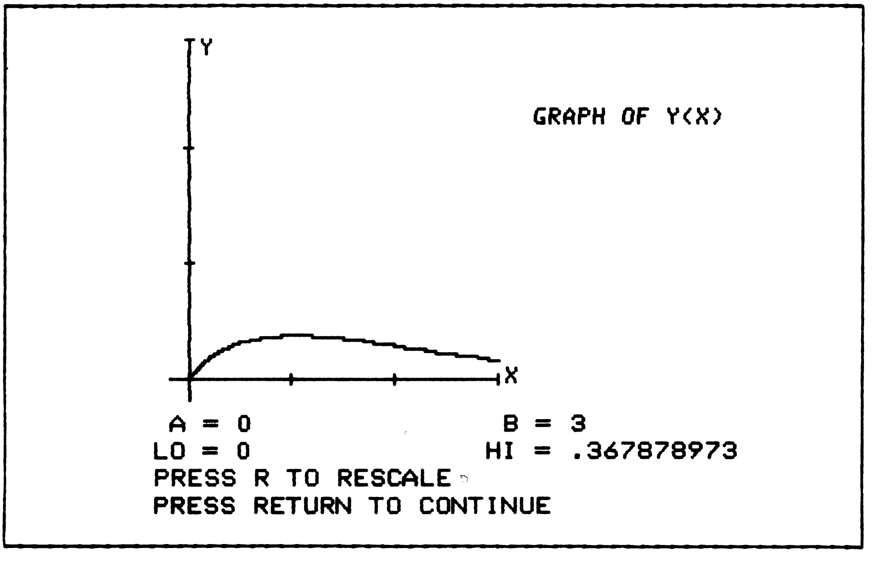
Solution. Start from the problem display, which will look like the one in Screen 1 if you have just completed Example 1. Press |C|. The cursor will jump to the first character in the current formula for Y', and the lines

|RETURN| ACCEPT ENTRY |ESC| ABORT ENTRY
ENTRY LIMIT: 100 CHARACTERS

will appear at the bottom of the screen. You may exit from change mode at any time by pressing |ESC|. Try it, press |C| again, and enter

Y' = EXP(-X) - Y A = 0 B = 3 Y(A) = 0 N = 30

Press |G| to execute the computation routine and display the output menu (same as Screen 2). Then press |S| to display the graph of Y shown in Screen 5.



Screen 5. The graph of the solution Y(X) computed for the problem $y' + y = e^{-x}$, $y(0) = 0$.

Press |R| to rescale the graph. Then press |D| to graph Y' (shown in Screen 6). Press |R| to rescale this graph as well, and then toggle between the graphs of Y and Y' by alternately pressing |S| and |D|.

Press |RETURN| to return to the output menu, and press |V| to examine the value tables for Y and Y'. Press |RETURN| to halt and continue the display, as desired.

The values of y(x) at x = 1, 2, and 3 are .367878973, .270670344, and .14936115. The true solution of this problem is $y = xe^{-x}$, and these representative values differ from the corresponding 9-place values of the true solution by .000000468, .000000222, and .000000551.

Examine also the graph and the table of values for y'(x). The derivative of the closed form solution of the problem is $y' = e^{-x}(1 - x)$, and the computed value -.0995740816 at x = 3 differs from the corresponding value of the true solution by .0000000551.

Example 3. Evaluate the definite integral

$$\int_0^1 e^{-x^2/2}\,dx.$$

Solution. Normally integrals with no closed form solution are best evaluated by integration routines, but we can approximate many definite integrals with the first order solver in the manner we now illustrate.

If the initial value problem

$$y' = e^{-x^2/2}, \quad y(0) = 0$$

is solved on the interval 0 to 1, then the computed value of the solution at x = 1 will approximate the given integral because

$$\int_0^1 e^{-x^2/2}\,dx = y(1) - y(0).$$

Since y(0) = 0, the value of the integral equals y(1) and is therefore approximated by the computed value at x = 1.

Return to the problem display and enter

Y' = EXP(-.5 * X * X)   A = 0   B = 1   Y(A) = 0   N = 20

When the entries have been checked, press |G| then |S| to solve. The computed value .855624394 at x = 1 can be obtained from the table of values for y (or in this case from the maximum value shown under the graph, since the integrand is positive). This value cannot be checked directly since it is impossible to solve this problem in closed form. However, the integrand is a well-known function: it is $\sqrt{2\pi}$ times the probability density function for the standard normal distribution (bell-shaped curve), whose values can be found in standard mathematical tables. If the computed value is divided by $\sqrt{2\pi}$, the quotient is .34144747 which, when rounded, agrees with the value .34134 taken from 5-place standard normal tables.

4. A NOTE ON APPLICABILITY

The Runge-Kutta method used in this program is by no means a universal solver for first order initial value problems. In fact, there are some relatively easy problems for which the method returns inaccurate results; most textbooks on numerical analysis contain examples to illustrate this point. In spite of these limitations and the need for caution, the method is a legitimate problem solving tool that provides insightful information in many problems, and its value to you will increase as you gain experience with it.

PROBLEMS

In Problems 1-10, find the computer solution to the initial value problem over the given interval.

1. $y' = x/7$, $a = 0$, $b = 4$, $y(0) = 1$, $n = 20$
2. $y' = x^2 + y^2$, $a = 0$, $b = 1$, $y(0) = 0$, $n = 10$
3. $y' = -y^2/x$, $a = 2$, $b = 4$, $y(2) = 2$
4. $y' = (x^2 + y^2)/2y$, $a = 0$, $b = 2$, $y(0) = 1$, $n = 20$

5. Same as problem 4, except $n = 40$

6. $y' = \cos x$, $a = \pi/6$, $b = \pi/2$, $y(\pi/6) = -\ln 2$, $n = 10$

7. $y' = -y^2e^x$, $a = 0$, $b = 2$, $y(0) = 1$

8. $xy' = x + y$, $a = 1$, $b = 4$, $y(1) = 0$, $n = 60$

9. $y' = x + y/x$, $a = 1$, $b = 4$, $y(1) = -3$, $n = 30$

10. $\theta r' = \cos\theta - r$, $a = \pi/6$, $b = 3\pi/2$, $y(0) = 3/\pi$, $n = 30$

11. In the problem of Example 1, about how large should you take n to find the true eight-place value, 5.38905610, of $y(1) = e^2 - 2$? Experiment to find out.

12. Resonance.

   a) Graph the solution of the initial value problem $y' = y/x + 3x\cos 3x$, $a = .1$, $b = \pi$, $y(.1) = 0$, $n = 30$. Rescale the graph.
   b) Repeat (a) with $b = 2\pi$.
   c) Repeat (a) with $b = 4\pi$ and $n = 100$.

13. a) $y' = \cos(x - y)$, $a = 0$, $b = 3$, $y(0) = 0$, $n = 30$
   b) Use the computer results to guess the true solution, and verify the suspected result.

14. a) $y' = (y^2\sin x)/(2y\cos x - 1)$, $a = -\pi/4$, $b = \pi/4$, $y(-\pi/4) = \sqrt{2}$, $n = 20$
   b) Clear the differential equation of fractions and show that the resulting equation is exact. Then solve the initial value problem directly.

15. Evaluate $\int_0^1 \sin(x^2)dx$ with (a) $n = 20$ and (b) $n = 40$.

16. Evaluate $\int_0^1 x^2e^{-x}\,dx$.

Y. Second Order Initial Value Problem

1. DESCRIPTION

This program provides numerical solutions to problems of the form

$$y'' = f(x,y,y'); \quad y(a) = y_0, \quad y'(a) = y'_0$$

for $y = y(x)$ and $y' = y'(x)$ over an interval $a \leq x \leq b$ with a fourth order Runge-Kutta method with up to one hundred steps. After entering f, a, b, y_0, y'_0, and the number of steps, you may elect to see graphs and to view and print value tables of both y and y'.

2. SECOND ORDER INITIAL VALUE PROBLEMS

The second order initial value problem

$$y'' = f(x); \quad y(a) = y_0, \quad y'(a) = y'_0$$

is usually studied early in calculus. In determining the position of a moving particle from its acceleration, initial position, and initial velocity, for example, the problem appears with x as time and y as position. The problem is then solved by integrating f(x) twice and substituting values given for y and y' at x = a to find the constants of integration.

Second order initial value problems appear also in the introduction to ordinary differential equations found in many

calculus books. Introductory presentations are often limited to problems of the form

$$py'' + qy' + ry = g(x); \quad y(a) = y_0, \quad y'(a) = y'_0,$$

where p, q, and r are constants and $p \neq 0$. The equation in this problem is linear with constant coefficients, and the explicit form of the function defining y'' is

$$f(x,y,y') = (g(x) - qy' - ry)/p.$$

One finds a closed form solution by first solving the equation with $g(x) = 0$, then finding a particular solution to the full equation, adding these solutions to form the general solution, and evaluating the two constants in the general solution.

For many problems, such as the ones that arise in describing the motion of an harmonic oscillator with an elementary forcing function, these techniques yield satisfactory results. However, there are many important problems for which other techniques are superior. One such technique is the application of numerical methods, with implementation on a computer.

The simplest numerical methods for second order initial value problems involve computing two tables of values,

$$w_0, w_1, \ldots, w_n \quad \text{and} \quad u_0, u_1, \ldots, u_n$$

corresponding to points

$$x_0, x_1, \ldots, x_n$$

in the given interval. The w_i approximate the values $y(x_i)$ of the function, and the u_i approximate the values $y'(x_i)$ of its first derivative.

The Toolkit program uses a fourth order Runge-Kutta solver to obtain the values w_i and u_i. Under this method, the points $x_0, x_1, \ldots, x_n$ are equally spaced across the interval. The distance between consecutive points is called the step size and is denoted by h:

$$h = (b - a)/n,$$

and the x_i are given by

$x_0 = a, \quad x_1 = a + h, \quad x_2 = a + 2h, \quad \ldots, \quad x_n = a + nh = b.$

The corresponding values of w_i and u_i are found as follows:

$$w_0 = y_0, \quad u_0 = y'_0,$$

and, for $i = 0, 1, \ldots, n - 1$,

$$w_{i+1} = w_i + h(u_i + (k_1 + k_2 + k_3)/6)$$
$$u_{i+1} = u_i + (k_1 + 2k_2 + 2k_3 + k_4)/6,$$

where

$$k_1 = hf(x_i, w_i, u_i)$$
$$k_2 = hf(x_i + h/2, w_i + hu_i/2 + hk_1/8, u_i + k_1/2)$$
$$k_3 = hf(x_i + h/2, w_i + hu_i/2 + hk_1/8, u_i + k_2/2)$$
$$k_4 = hf(x_i + h, w_i + hu_i + hk_3/2, u_i + k_3).$$

The method is relatively accurate, with error of order h^5 at each step, and h^4 across the interval.

3. STEP BY STEP

Load the program from the menu, read the greeting messages, and continue on to the problem display in Screen 1, which sets the stage for the first example.

```
EQUATION AND ENDPOINTS

  Y" = -4 * Y
   A = 0                    B = 3.14159265

INITIAL VALUES

Y(A) = 0               Y'(A) = 1

NUMBER OF STEPS             N = 20

|C|HANGE ENTRY    |G|O ON    |Q|UIT
```

Screen 1. The problem display.

Example 1. Solve the initial value problem

$$y'' = -4y, \quad y(0) = 0, \quad y'(0) = 1$$

on the interval from 0 to π, using $n = 20$ steps.

Solution. The equation and endpoints from this problem are the ones in the initial problem display in Screen 1. When you have reviewed the entries, press |G| to execute the solution routine. In about four seconds, the computer will present the output menu shown in Screen 2.

```
                    OUTPUT MENU

      |S| .. SOLUTION:   GRAPH OF Y(X)
      |D| .. DERIVATIVE:   GRAPH OF Y'(X)
      |V| .. VALUES OF Y(X) AND Y'(X)
      |H| .. HARD COPY OF VALUES
      |P| .. PROBLEM DISPLAY
      |Q| .. QUIT

      PRESS LETTER OF YOUR CHOICE  |_|
```

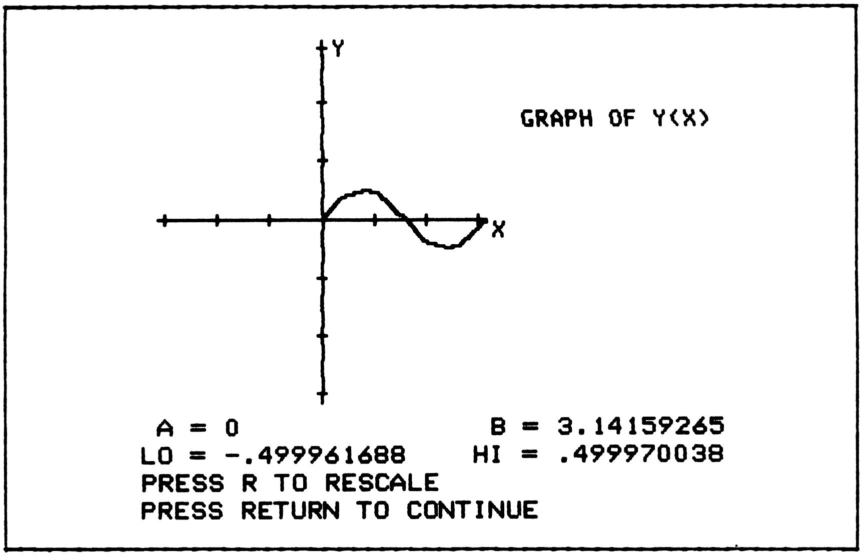
Screen 2. The output menu.

Press |S| to graph the solution computed for Y(X), shown in Screen 3, and press |R| to rescale.

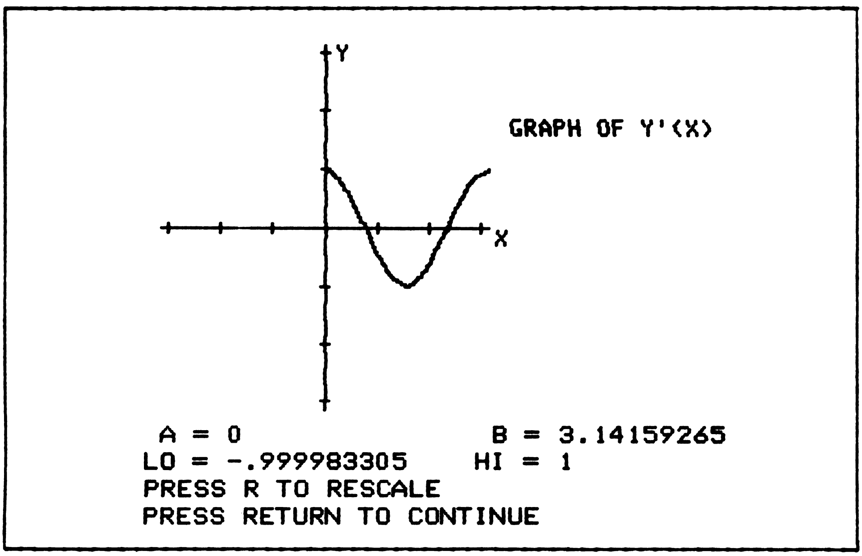
The LO and HI in Screen 3 are the minimum and maximum values computed for y on the interval [a,b].

The graph in Screen 3 suggests that the exact solution is $y = .5 \sin 2x$, as can be readily verified by substituting the expression and its second derivative in the equation $y'' = -4y$ and checking the initial conditions. From expression $y = .5 \sin 2x$ we find that the high value computed for y at $x = 3\pi/4$ is in error by less than .00003.

Now press |D| to display the graph of Y'(X) shown in Screen 4, and press |R| to rescale. The graph shows a full period of the exact derivative $y' = \cos 2x$ along with the minimum and maximum values computed over the interval. After viewing the graph, press |S| and |D| alternately to "toggle" between the graphs of Y and Y'.



Screen 3. The graph of the solution Y(X) computed for the problem $y'' = -4y$, $y(0) = 0$, $y'(0) = 1$.



Screen 4. The graph of the values computed for Y'(X).

Press |RETURN| to return to the output menu, and press |V| to list the values computed for Y and Y'. You may halt and continue the listing as desired by pressing |RETURN|. The error in the value computed for y at $x = \pi$ is

$$|-9.4500E\text{-}05 - .5 \sin 2\pi| = 9.4500E\text{-}05,$$

a reasonable result in view of the relatively large step size of $\pi/20$. The value .999966601 computed for $y' = \cos 2x$ at $x = \pi$ is in error by less than four parts in a hundred thousand.

With n = 40 steps (and very little more computing time) we would find $y(\pi) = 5.96265136\ E\text{-}06$ and $y'(\pi) = .999998956$, which are definite improvements over the results for n = 20.

If your computer is connected to a printer and you know the slot number of the interface card (the usual number is 1), you may print the values of Y and Y'. To do so, press |H| from the output menu, press the slot number, and press |RETURN| to start printing.

Now return to the output menu, if you are not there already, and press |P| for the problem display to prepare for the next example.

Example 2. Solve the initial value problem

$$y'' + y' + y = (x - 1)e^{-x}, \quad y(0) = 0, \quad y'(0) = 1$$

on the interval 0 to 3, using n = 30 steps.

Solution. Start from the problem display, which will look like the one in Screen 1 if you have just completed Example 1. Press |C| for change mode. The cursor will jump to the first character in the current formula for Y', and the lines

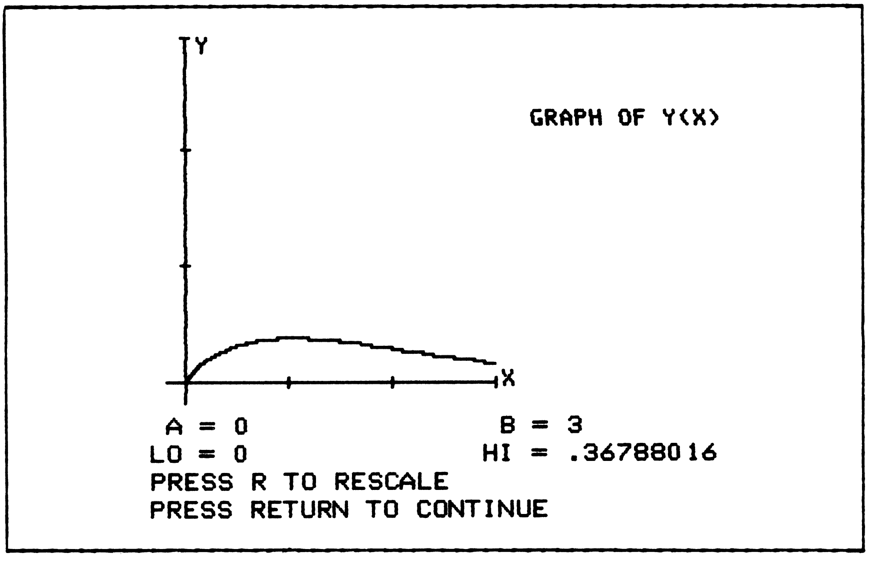
USE |Z| FOR THE VARIABLE Y'
|RETURN| ACCEPT ENTRY |ESC| ABORT ENTRY
ENTRY LIMIT: 100 CHARACTERS

will appear at the bottom of the screen. You may exit from change mode at any time by pressing |ESC|. Try it, press |C| again, and, using Z for the variable Y', enter

Y" = -Z - Y + (X - 1)*EXP(-X)

A = 0   B = 3   Y(A) = 0   Y'(A) = 1   N = 30

After checking your entries, press |G| to execute the computation routine and display the output menu (same as Screen 2). Then press |S| to display the graph of Y shown in Screen 5.



Screen 5. The graph of the solution Y(X) computed for the problem $y'' + y + y' = (x - 1)e^{-x}$, $y(0) = 0$, $y'(0) = 1$.

Screen 5 shows the maximum computed value HI = .36788016. The exact solution to this problem is $y = xe^{-x}$, and at $x = 1$ this function achieves a maximum of $1/e$, which is .36787944 to 8 places.

Now graph y', and note, for example, that this display reflects the behavior of y relative to the increase or decrease in values. The minimum value computed for y' is -.13533524. The derivative of the exact solution is $y' = (1 - x)e^{-x}$, and the computed minimum value compares well with the true eight-place value $-e^{-2} = -.13533528$.

For another measure of accuracy, examine the tables of values for y and y', checking especially the values at x = 3. The computed value of y at x = 3 is .149362123, which differs by .000000918 from .149361205, the nine-place value of the exact solution. The corresponding difference for y' is .000000659.

Example 3. Solve the initial value problem

$$y'' = y(x^2 - 1), \quad y(0) = 1, \quad y'(0) = 0$$

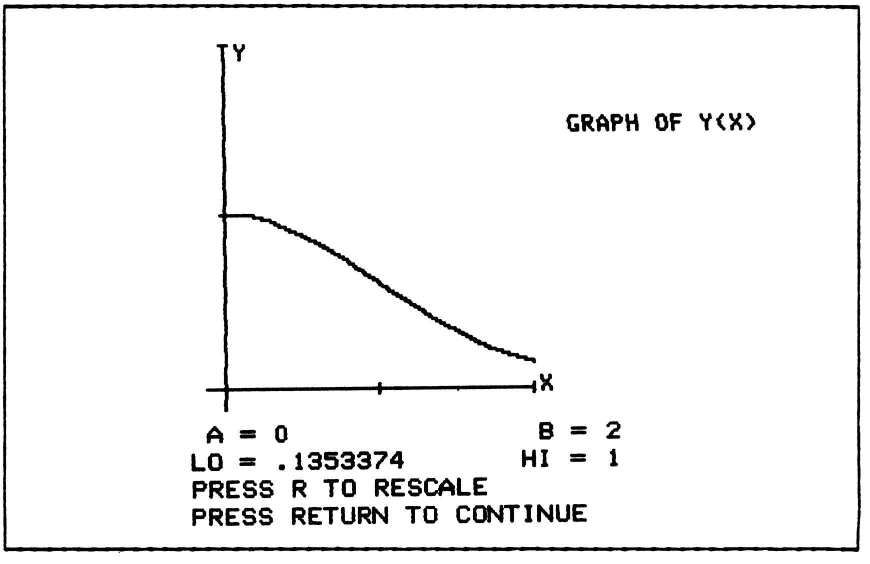
on the interval 0 to 2, using 20 steps.

Solution. Go to the problem display to enter

Y" = Y * (X * X - 1) A = 0 B = 2

Y(A) = 1 Y'(A) = 0 N = 20

Then press |G| and |S| to graph the solution (Screen 6).



Screen 6. The graph of the solution Y(X) computed for the problem $y'' = y(x^2 - 1)$, $y(0) = 1$, $y'(0) = 0$.

The display in Screen 6 shows a curve starting from $y = 1$ with slope 0 at $x = 0$, then descending to .1353374 at $x = 2$. The graph suggests an inflection point near $x = 1$.

The computer solution agrees quite well with the exact solution, which is $y = \exp(-x^2/2)$. Those who have studied probability or statistics may recognize this expression as the standard normal probability density function multiplied by $\sqrt{2}\pi$, and the graph as a segment of the standard bell-shaped curve. As a measure of accuracy, compare the value computed at $x = 2$ with the 7-place value of e^{-2}, which is .1353353: the difference is .0000021.

Press |D| to continue on to graph y', noting the excellent agreement of the computed solution with the derivative of the true solution, $y' = -x \exp(-x^2/2)$. You may also wish to examine the tables of values to complete this example.

4. ON USING NUMERICAL METHODS

The Runge-Kutta method used in this program yields reasonable solutions for many, but by no means all, second order initial value problems. Just as closed form methods fail for some problems, Runge-Kutta methods also fail for some of the same ones, and for some different ones, too. When these methods fail, a variety of other numerical methods, as well as series and transform techniques, may apply. The compleat problem solver is prepared with a good repertoire of methods, of which the present Runge-Kutta method is but one.

PROBLEMS

Solve each initial value problem over the interval [a,b] with the given step size n.

1. $y'' = -y' - 5y$; $y(0) = 4$, $y'(0) = -2$, $a = 0$, $b = 8$, $n = 40$

   Exact solution: $y = 4e^{-x/2} \cos 2x$.

2. $y'' + 4xyy' + 2y^2 = 0$; $y(0) = 1$, $y'(0) = 0$, $a = 0$, $b = 3$, $n = 15$

   Exact solution: $y = 1/(1 + x^2)$.

3. $y'' + 16y = 8 \cos 4x$; $y(0) = 0$, $y'(0) = 0$, $a = 0$, $b = 4$, $n = 40$

   Exact solution: $y = x \sin 4x$.

4. $y'' + xy' + 2y = 0$; $y(0) = 0$, $y'(0) = 1$, $a = 0$, $b = 3$, $n = 20$

   Exact solution: $y = xe^{x^2/2}$.

5. Repeat Problem 4 with $n = 10$.

6. $y'' + 4y' + 4y = xe^{-x}$; $y(0) = 1$, $y'(0) = 1$, $a = 0$, $b = 2$, $n = 10$

   Exact solution: $y = (1 + 3x + x^3/6)e^{-x}$.

7. $x^2y'' + 6xy' + 6y = 12$; $y(1) = 4$, $y'(1) = 1$, $a = 1$, $b = 4$, $n = 30$

   Exact solution: $y = 5x^{-2} - 3x^{-3} + 2$.

8. $x^2y'' + 5xy + 4y = 0$; $y(1) = 1$, $y'(1) = -1$, $a = 1$, $b = 5$, $n = 30$

   Exact solution: $y = (1 + \ln x)/x^2$.

9. $x^2y'' + 3xy + y = 0$; $y(1) = 0$, $y'(1) = 1$, $a = 1$, $b = 4$, $n = 30$

   Exact solution: $y = (\ln x)/x$.

10. $16y'' + 8y' + 145y = 0$; $y(0) = 1$, $y'(0) = 1$, $a = 0$, $b = 4$, $n = 20$

    Exact solution: $y = e^{-x/4}(\cos 3x + (5/12) \sin 3x)$.

11. $(1 + x^2)y'' - xy' + 4y = 0$; $y(0) = 1$, $y'(0) = 0$, $a = 0$, $b = 3$, $n = 30$

    Series solution: $y = 1 + \sum_{n=1}^{\infty} = \frac{4(1 + (n - 1)^2)}{(2n)!} x^{2n}$.

Z. Damped Oscillator

1. DESCRIPTION

This program allows you to explore solutions of the equation

$$X''(T) + GX'(T) + (W_0)^2 X(T) = 0.$$

After entering the coefficients, you may elect to see the graphs of X(T) (displacement vs. time) and X'(T) (velocity vs. time), and the phase plane plot of velocity vs. displacement. You may also graph the energy $E = (X')^2 + W_0{}^2 X^2$ vs. time, and request the numerical values of the displacement, velocity, and energy for different times.

2. DAMPED OSCILLATORS

A simple harmonic oscillator is a mass that moves along a straight line under the action of a force

$$F = -kX$$

that is opposite in direction and proportional in magnitude to the mass's displacement.

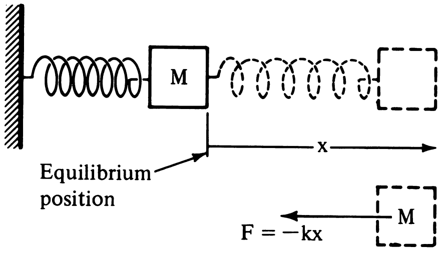


Fig. 1. Model for periodic motion. If the spring obeys Hooke's law and there is no retarding force, the motion is simple harmonic motion.

If we write

$$F = mA(T),$$

the equation

$$A(T) = -\frac{k}{m}X(T) \tag{1}$$

describes the motion. Since the acceleration A(T) of the mass is the second derivative of the displacement with respect to time, Eq.(1) can be rewritten as

$$X''(T) = -\frac{k}{m}X(T)$$

or

$$X''(T) + \frac{k}{m}X(T) = 0. \tag{2}$$

Equation (2) is a differential equation for the displacement X(T) of the mass.

If the oscillating mass is subject not only to restoring force that is proportional to displacement but also to a retarding force such as friction or viscous drag (the block in Fig. 1 scrapes on the floor, or is immersed in a fluid), then the retarding force will damp out the motion after the system has been started and left to run. Such an oscillator is called a <u>damped oscillator</u>.

The simplest form of retarding force, and the one that leads to the simplest mathematical model of an oscillator with damping, is one in which the retarding force is opposite in direction and proportional in magnitude to the velocity:

$$\text{Retarding force} = -bV(T), \quad b > 0.$$

Newton's second law, F = mA, then gives

$$-bV(T) - kX(T) = mA(T),$$

or

$$A(T) = -\frac{b}{m}V(T) - \frac{k}{m}X(T), \tag{3}$$

as an equation for the motion. Since the velocity and the acceleration are the first and second derivatives of the displacement with respect to time, Eq.(3) can be written as

$$X''(T) + \frac{b}{m}X'(T) + \frac{k}{m}X(T) = 0. \tag{4}$$

Like Eq.(2), Eq.(4) is a differential equation for the displacement X(T) of the mass. Equation (2) is Eq.(4) with b = 0.

The quantity b/m is frequently represented by the Greek letter gamma, but in this program it appears as G. The quantity k/m is also w_0^2, the square of the natural angular frequency of the oscillator, and in this program it appears as $(WO)^2$. With these notational changes, Eq.(4) becomes

$$X''(T) + GX'(T) + (WO)^2X(T) = 0. \tag{5}$$

The program enables you to explore the solutions of Eqs.(2) and (4) and the ways in which they depend on the initial values of the displacement and the velocity.

Even if you have learned how to solve Eqs.(2) and (4), you can use this program to examine many more cases than you could reasonably examine if you had to plot the solutions by hand. In particular, the program can help you to review and learn more about

simple pendula
masses on springs
simple harmonic motion
period, frequency, angular frequency
amplitude
conservation of energy

3. STEP BY STEP

After loading the program, read the greeting messages and continue on to the input menu, shown in Screen 1.

```
X" + GX' + (WO)^2X = 0
 G = 0                WO = 1.5708
XO = 24               VO = 0

|D|ISPLACEMENT:  X VS T
|V|ELOCITY:      V VS T

|P|HASE PLANE:   V VS X
|E|NERGY VS T

|O|RDINATES OF X, V, E
|N|EW SCREEN

|R|ESCALE GRAPH
|C|HANGE VALUES

|Q|UIT

|RETURN| TO PLOT |_|
```

Screen 1. The input menu.

The letters on the left side of the Screen 1 are abbreviations for the available operations. Press

- |D| to display displacement vs. time.
- |V| to display velocity vs. time as a graph working its way down from a horizontal line that represents the initial energy of the system.
- |P| (for phase plane) to display velocity vs. displacement.
- |E| to display the energy vs. time.
- |O| (for ordinates) to obtain numerical values of time, displacement, velocity, and energy (in ergs, or units of gm cm^2/s^2). A vertical line is plotted at a small

value of the time. Press |->| and |<-| to move the vertical line (initially concealed by the vertical axis) to different values of the abscissa. The values for T, X, V, and E will appear at the bottom of the screen. You may stop the ordinate presentation and return to graphics mode at any time by pressing |ESC|. You may return to the input menu at any time by pressing |RETURN|.

|R| to rescale.
|N| for a new screen.
|C| to change the oscillator.
|Q| to quit the program.

Special command features. You need not return from the graphics screen to the input menu to execute menu options. You may overlay graphs or clear the screen between displays by commands directly from the graphics screen. You may also stop a plot in progress by pressing |ESC|.

Example 1. A body of mass 10 gm moves with simple harmonic motion of amplitude 24 cm (maximum $|x|$) and period 4 s. The position coordinate is +24 cm (its maximum) when T = 0. Compute

a) the position of the body at T = 0.5 s
b) the magnitude and the direction of the force acting on the body at T = 0.5 s
c) the minimum time required for the body to move from its initial position to the point where X = −12 cm
d) the velocity of the body when X = −12 cm

Then plot the energy of the system and examine the phase plane.

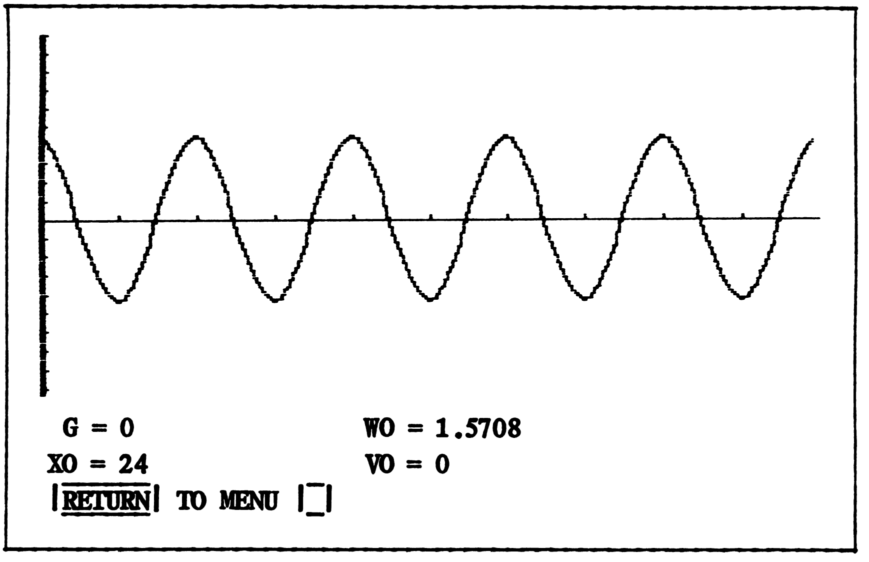
Solution. The values of G, WO, X(0), and V(0) for this problem are the ones currently entered in the input menu. The value of WO is determined by observing that

$$WO = \frac{2\pi}{\text{period}}$$

and the period is 4 s:

$$WO = \frac{1.5708}{\text{seconds}}.$$

The first part of the problem we are solving asks for the position of the body at T = 0.5 s. Display the displacement vs. time by pressing |D|. The computer will draw the plot shown in Screen 2.



Screen 2. Undamped X vs. T.

The program initially chooses the time scale to display 5 periods of the natural angular frequency. It will automatically clear the screen and rescale the axis whenever you rescale the time or choose a new WO.

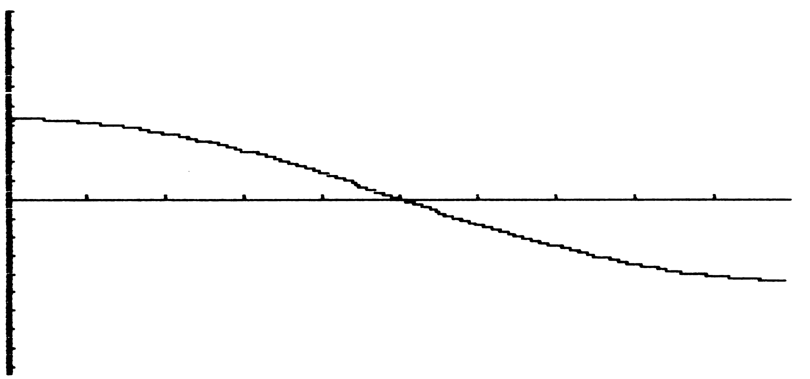
To find the position of the body at T = 0.5 s, press |O| for ordinate and then press |->| until the numerical display at the bottom of the screen shows T = .5. The problem here, we soon find, is that the T-values skip from .4 to .6 as they increase from the starting time to T = 0. To find the value of X(T) when T = .5 we must rescale.

Press |ESC| to leave ordinate mode, and then press |R|. The message at the bottom of the screen will change to

```
TMAX = |2|0                 XMAX = 55.0597
VMAX = 86.4876              EMAX = 1421.22
|RETURN| ACCEPT ENTRY       |ESC| ABORT ENTRY.
```

The numbers here are the maximum values the computer is currently showing on the coordinate axes. Do not confuse these values with the maximum values of the variables in the oscillating system. For example, the oscillator we are studying has an amplitude of X = 24 cm. The displayed XMAX = 55.06 tells us that the vertical axis on the screen reaches 55.06 cm. We therefore expect the graph of X(T) to go about halfway up, as it does in Screen 2.

Since we wish to change the time scale, press |2| |RETURN|. This erases the old graph as well; the screen is cleared automatically whenever TMAX or WO is changed. Press |ESC| to leave input mode. You may now replot the displacement vs. time by pressing |D| to obtain the display in Screen 3.



```
G = 0                       WO = 1.5708
XO = 24                     VO = 0
|RETURN| TO MENU |_|
```

Screen 3. The graph of X(T) with $0 \leq T \leq$ TMAX = 2.

If we now press |O| and then |->|REPT| to move the vertical line, we can see from the data at the bottom of the screen that X is approximately 17 cm when T = .5.

The second part of the problem asks us to compute the force on the body at T = 0.5 s. We know that at T = 0.5 s the displacement is 17 cm, and that the force is given by

$$F = -kX.$$

To compute F, we must first determine the value of k. We know that

$$(WO)^2 = k/m$$

and that WO = 1.5708 and m = 10 gm. Thus

$$k = m(WO)^2 = 10(1.5708)(1.5708) = 24.67 \text{ dyne/cm.}$$

When X = 17 cm, the force is

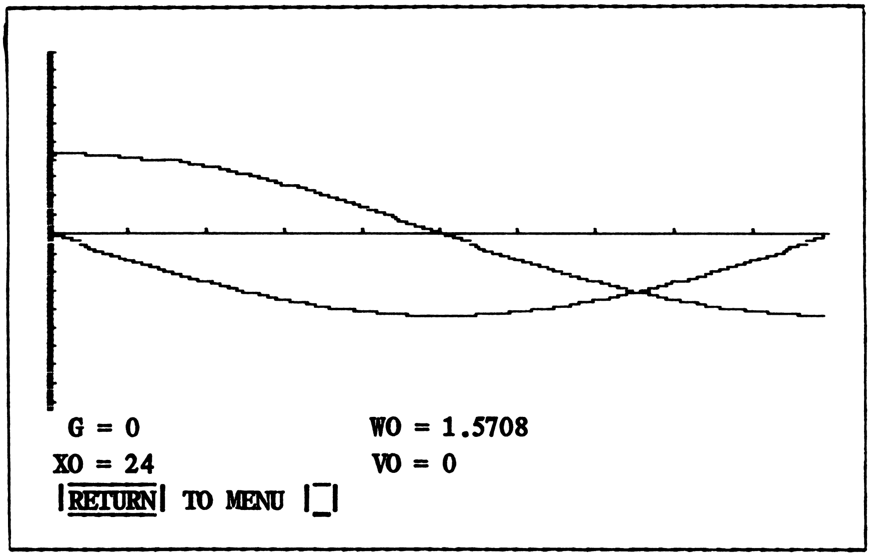
$$F = -(24.67)(17) = -419.5 \text{ dyne.}$$

The third part of our problem asks for the minimum time for the body to move from its initial position to the point where X = −12 cm. By pressing |->|REPT| to move further along the time axis on the ordinate screen, we see that X reaches the value of −12 cm at about 1.34 seconds. At that time the velocity is approximately −32.4 cm/sec.

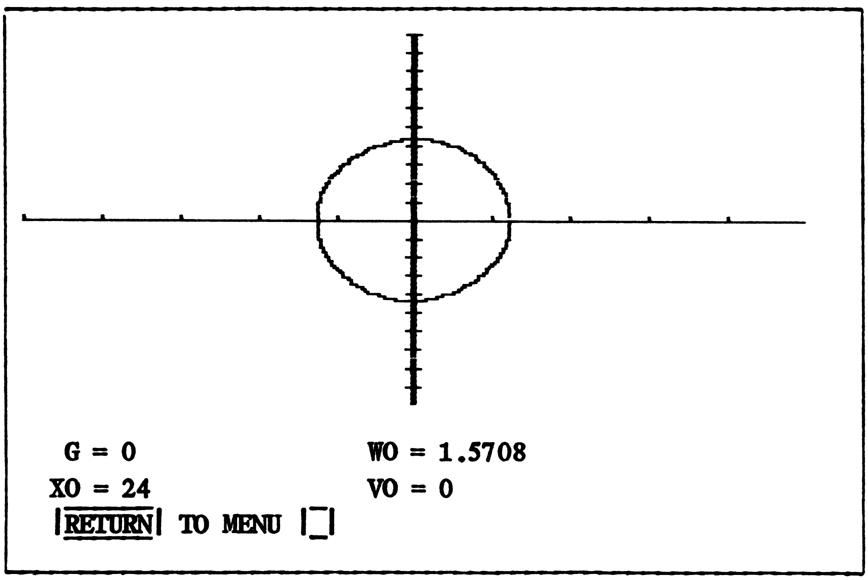
Now press |ESC| and |V| to plot the velocity V(T) shown in Screen 4.

Press |E| to plot the energy vs. time. The graph in this case is the horizontal line that represents the initial energy of the system. (If you tire of watching the plot, press |ESC|.) There is no retarding force like friction to drain energy away, there is no external force to add energy, and G = 0.

Rescale the time so that T MAX = 4 and plot the displacement vs. velocity by pressing |N| |P|. The completed plot will look like the one in Screen 5.



Screen 4. X(T) and V(T) together for $0 \leq T \leq$ TMAX = 2.



Screen 5. The phase plane, V(T) vs. X(T), $0 \leq T \leq 4$.

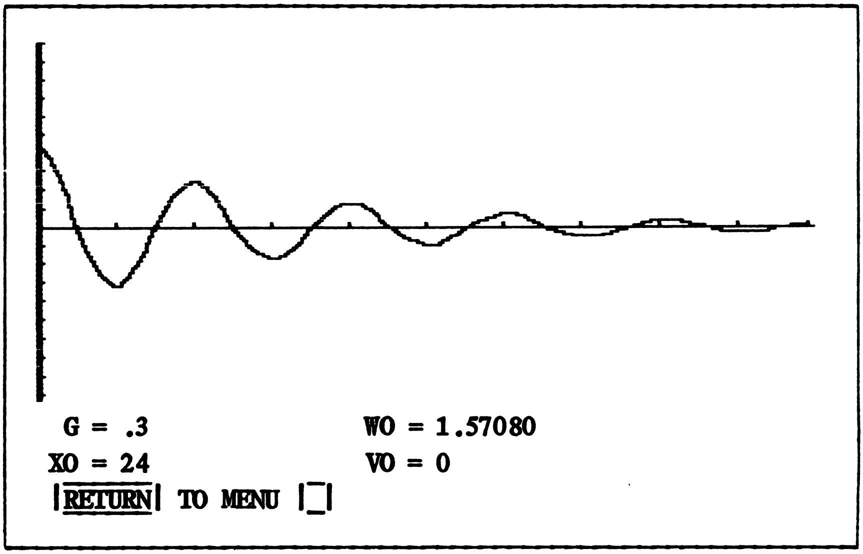
The phase plane is a closed curve because

$$V(T)^2 + (WO)^2 X(T)^2 = \text{const.} \tag{6}$$

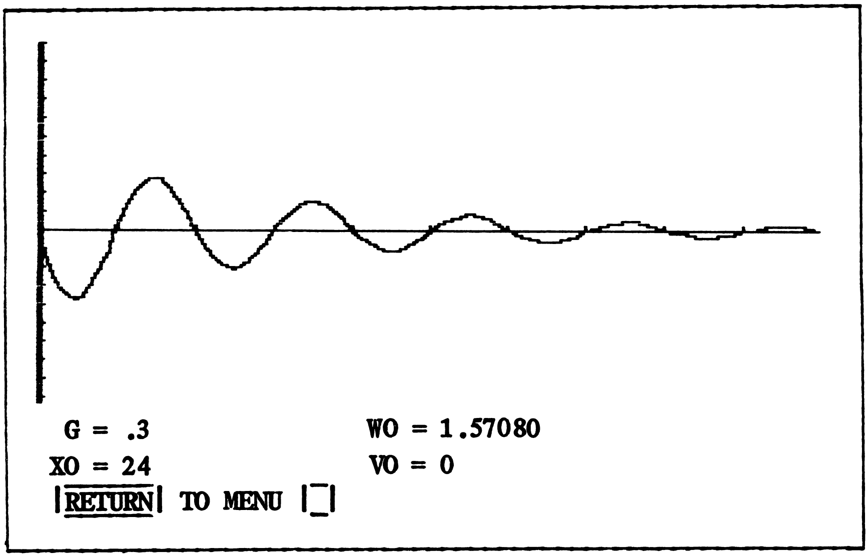
for the system under study (see Problem 10). Thus, the graph of V vs. X is an ellipse. From a physical point of view we expect the graph to be a closed loop because the energy of the system is constant.

Example 2. Add a retarding force with G = .3 to the oscillator of Example 1. Graph X(T), V(T), and E(T) and then plot V vs. X in the phase plane.

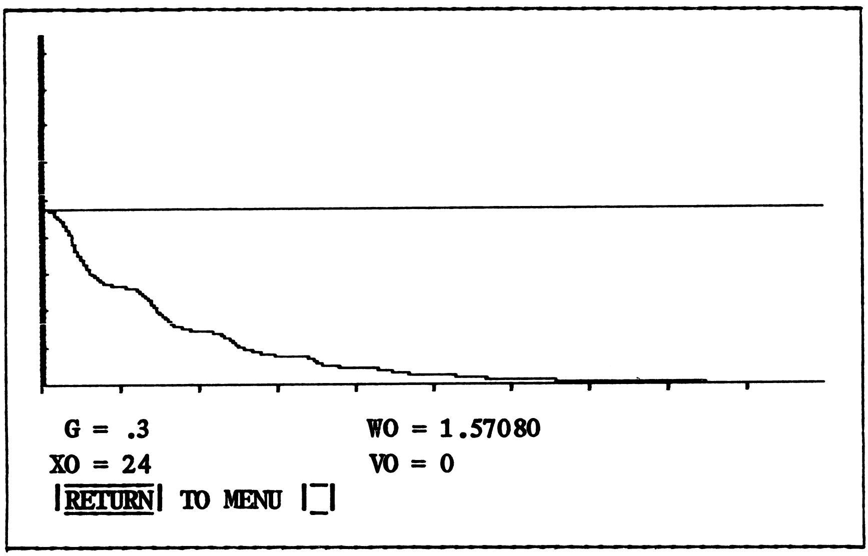
Solution. Press |RETURN| for the input menu, and press |C| |.| |3| |RETURN| |ESC| to enter G = .3. Then press |R| to rescale time and enter TMAX = 20. Next press |D| to graph the displacement X(T), shown in Screen 6. After that, press |N| |V| to graph the velocity (Screen 7). Finally, press |N| |E| to graph the energy (Screen 8).



Screen 6. X(T) when G = .3, 0 ≤ T ≤ 20.



Screen 7. V(T) when G = .3, 0 ≤ T ≤ 20.



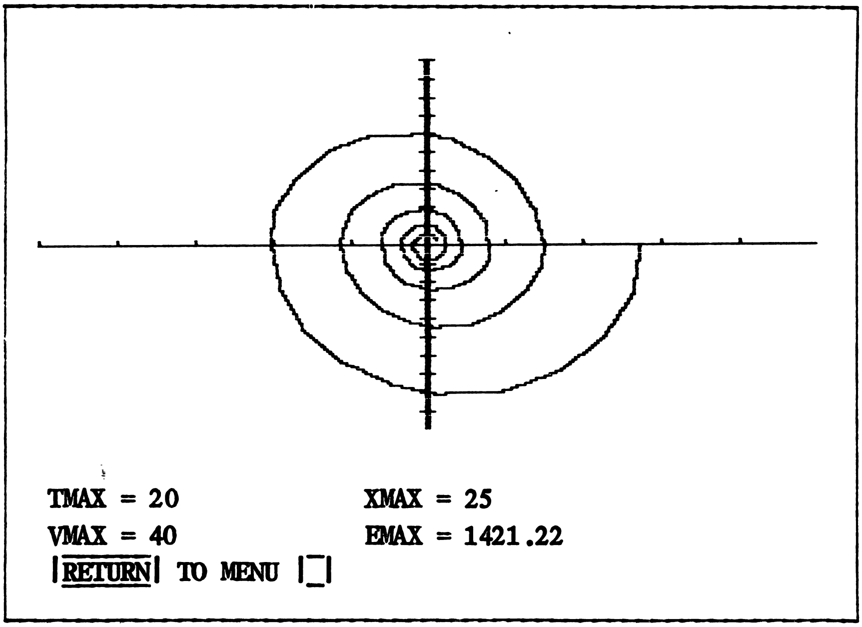
Screen 8. E(T) for G = .3, 0 ≤ T ≤ 20.

As time passes, energy is drained from the system and the energy drops toward zero from its initial level. Note the step-like character of the energy curve. The slope (rate at which energy is lost) varies with the velocity, being greatest when the mass is moving fast, and least when V is near zero. (See Problem 11.)

Before graphing V vs. X in the phase plane, press |R| and rescale the display by entering

XMAX = 25   VMAX = 40

Then press |N| and |P|. The resulting display is shown in Screen 9.



Screen 9. The phase plane plot, V vs. X, shows V and X decreasing with time as energy is drained from the oscillating system.

PROBLEMS

1. What physical principle is illustrated by the fact that the plot of displacement vs. velocity is a closed curve when there is no damping? Is your answer consistent with what you have seen the displacement vs. time curve to be in a case in which there is no damping?

2. If you look carefully at a plot of energy vs. time in a lightly damped (G small compared to WO) system, you will see that there are regions in which the energy vs. time curve is very nearly horizontal (i.e., parallel to the time axis). Explain this phenomenon physically.

3. For a fixed value of WO, find the period of the oscillator as a function of G. Plot the frequency of the damped oscillator as a function of G/2WO. Plot the frequency of the damped oscillator as a function of (G/2WO)∧2.

4. When an oscillator is damped, its angular frequency W is less than the "natural" angular frequency WO that the oscillator would have with the same m and F if no damping were present. The two frequencies are related by the equation

$$W^2 = WO^2 - G^2/4.$$

   a) What is the frequency of the oscillator when G = 2WO?
   b) Try various cases, like G = 4 and WO = 2, in the oscillator described in the text.

5. For a fixed set of initial conditions, examine the behavior of the solutions for:
   G > 2WO (solution oscillates)
   G = 2WO (critical damping)
   G < 2WO (overdamped).
   Give a physical explanation of each behavior.

6. Show that if G is small (compared to what?) then the energy in the system goes down exponentially.

7. How can you estimate the values of G and WO for a playground swing?

8. How can you estimate the values of G and WO for a tuning fork?

9. If you choose a negative value for G, all the variables behave oddly. Give a physical explanation for what the mathematical model "thinks" is going on.

10. Verify Eq.(6) by differentiating its left-hand side with respect to T, factoring out V = X(T), and showing that the remaining factor is the left-hand side of Eq.(5) with G = 0. In mathematical terms, Eq.(6) holds when G = 0 because X'(T) is then an integrating factor for Eq.(5).

11. Enter G = .3, WO = 2, XO = 24, and VO = 0 and graph the energy E(T). Add the graph of V(T) to the display to see the relation between V and the slope of E(T). Energy is lost more rapidly when |V| is large than it is when V is close to zero.

Control-Z. Forced Damped Oscillator

1. PURPOSE

This program enables you to investigate the behavior of forced damped oscillators.

2. DESCRIPTION

In this program the oscillator of Program Z is driven by an external force $(f/m)\cos(\omega t)$ that varies periodically in time with an angular frequency ω. The oscillator's equation of motion is

$$X''(t) + \frac{b}{m}X'(t) + \frac{k}{m}X(t) = \frac{f}{m}\cos(\omega t).$$

In the program it appears as

$$X'' + GX' + (WO)^2X = F\,COS\,(WT).$$

The quantity $F = f/m$ is the driving force per unit mass.

The behavior of a forced damped oscillator is more complex (and interesting) than that of an undriven oscillator. This program will help you to review and learn more about simple harmonic motion; period, frequency, angular frequency, amplitude; conservation of energy; forced oscillations; resonance; homogeneous and nonhomogeneous second order linear differential equations with constant coefficients.

3. RELATION TO "DAMPED OSCILLATOR"

The present program is similar to DAMPED OSCILLATOR in form as well as content. Basically, DAMPED OSCILLATOR may be regarded as FORCED DAMPED OSCILLATOR with F set equal to 0. We shall assume that you have already worked through the examples in Chapter Z.

4. STEP BY STEP

Load the program by holding down the CTRL key and pressing |Z|. (Do not press |Z| first, for this will indicate the preceding program, DAMPED OSCILLATOR.) Read the greeting messages and go on to the input menu, shown in Screen 1.

$$X'' + GX' + (WO)^2X = F\ COS(WT)$$

```
G = 1            WO = 2          F = 40
W = 4            XO = 24         VO = 0

|D|ISPLACEMENT:  X VS T
|V|ELOCITY:      V VS T
|P|HASE PLANE:   V VS X

|E|NERGY VS T
|F|FORCING FUNCTION VS T

|T|RANSIENT VS T
|S|TEADY STATE VS T
|B|OTH

|O|RDINATES OF X, V, E
|N|EW SCREEN
|R|ESCALE GRAPH

|RETURN| TO PLOT   |C|HANGE VALUES   |Q|UIT   |_|
```

Screen 1. The input menu.

As in the program DAMPED OSCILLATOR, G measures the damping in the system and WO is the natural angular frequency of the oscillator. The quantity F is the amplitude of the driving force per unit mass (dimensions in mks are newtons/kilogram) and W is the angular frequency of the driving force.

The program opens with G, WO, F, W, XO, and VO set for Example 1.

The letters on the left side of the screen are abbreviations for the operations now available to you. To select one of these options, press

|D| to display displacement vs. time
|V| to display velocity vs. time
|P| (for phase plane) to display velocity vs. displacement
|E| to display the energy E vs. time
|F| to display the driving force vs. time
|T| for the transient mode
|S| for the steady state mode
|B| for both transient and steady state combined. This is the default mode of the program.
|O| (for ordinates) to obtain numerical values of time, displacement, velocity, and energy. A vertical line is plotted at a small value of time. Press |->| and |<-| to move the vertical line (initially concealed by the vertical axis) to different values of the abscissa. A table of values for T, X, V, and E will appear across the bottom of the screen. You can leave the ordinate presentation via |ESC| or |RETURN|, the latter returning you to the input menu.
|R| to rescale.
|N| to clear the screen.
|C| to change parameter values.
|Q| to quit the program.

As with DAMPED OSCILLATOR, graphs can be displayed on the screen together to show relationships among variables.

The program automatically chooses the initial time scale to display 5 periods of the natural angular frequency.

It will automatically clear the screen and rescale the axis whenever you rescale time or choose a new WO.

Example 1. Investigate the behavior of the forced damped oscillator if

$$G = 1 \quad WO = 2 \quad F = 40 \quad W = 4 \quad X0 = 24 \quad V0 = 0$$

Scale the display axes with

$$TMAX = 10 \quad XMAX = 30 \quad VMAX = 10 \quad \text{and} \quad EMAX = 1500$$

Solution. We are asked to investigate the behavior of the oscillator for the first ten seconds. From a mathematical point of view we are being asked to investigate the solution of the equation

$$X'' + X' + 4X = 40\,COS(4T), \quad 0 \leq T \leq 10$$

subject to the initial conditions

$$X0 = 24 \quad \text{and} \quad V0 = 0.$$

The program enables us to investigate physical properties of the oscillator as well.

Press |R| and enter the suggested maximum values of T, X, V, and E. When the values have been entered, press |D| to graph the displacement X(T) (Screen 2).

The solution X(T) shown in Screen 2 is the sum of two functions, $X_h(T)$, a solution of the homogeneous equation

$$X'' + X' + 4X = 0,$$

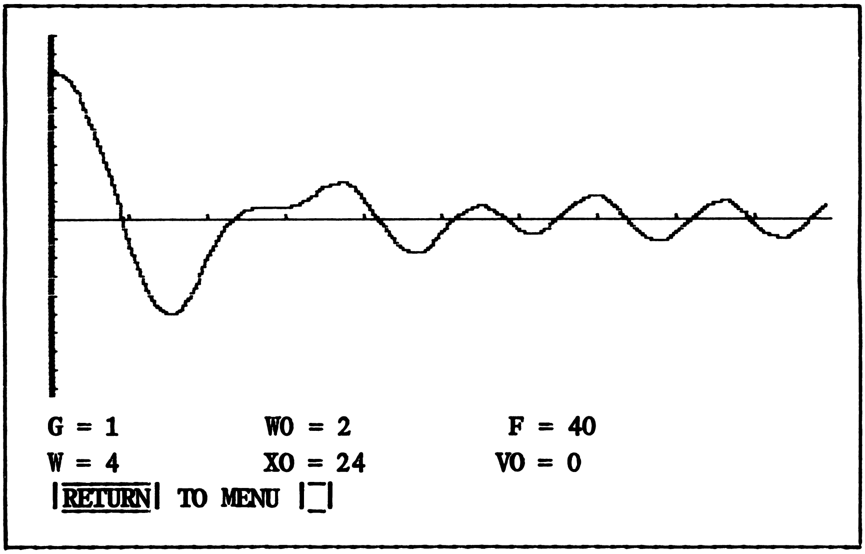
and $X_p(T)$, a particular solution of the equation

$$X'' + X' + 4X = 40\,COS(4T).$$

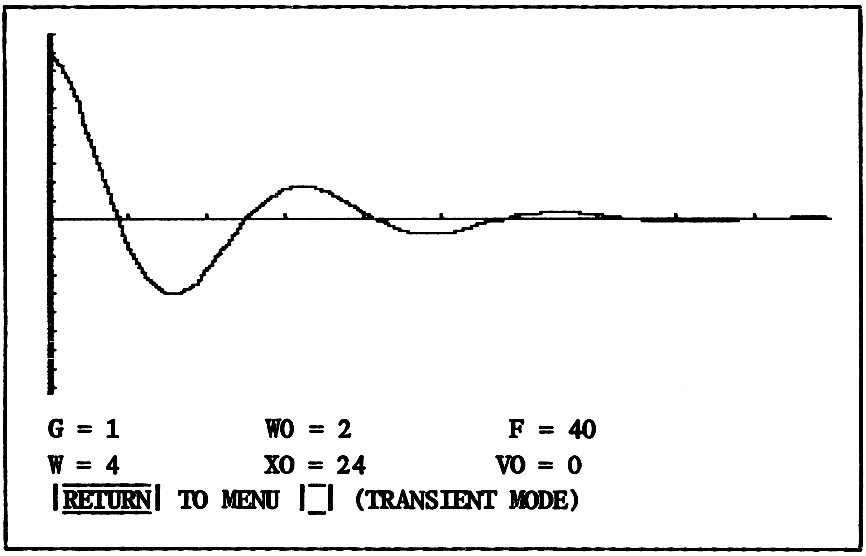
The constants in $X_h(T)$ have been chosen to make the sum

$$X(T) = X_h(T) + X_p(T)$$

satisfy the initial conditions of the problem (more about this in a moment).



Screen 2. The graph of X(T), $0 \le T \le 10$.



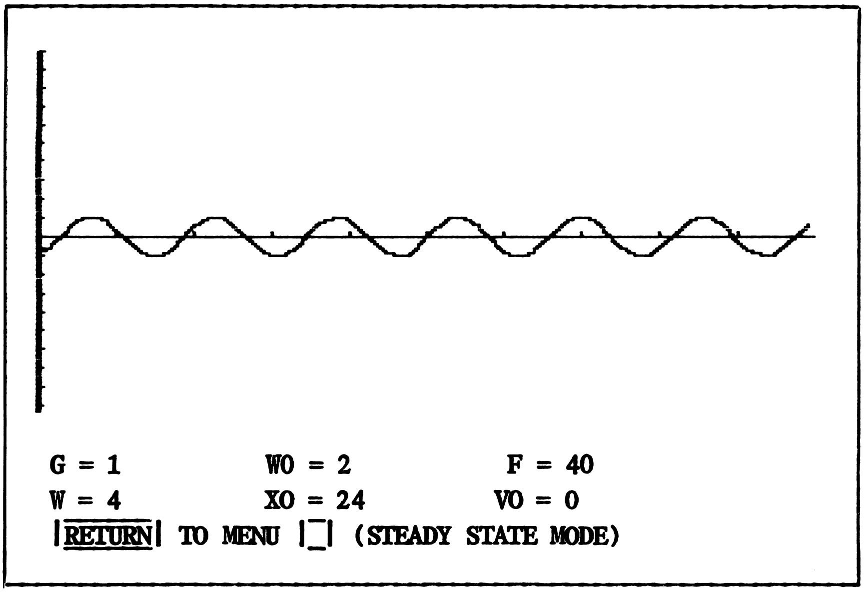
Screen 3. The graph of $X_h(T)$, the homogeneous or transient part of X(T), $0 \le T \le 10$.

Now press |T| for the transient mode, |N| for a new screen, and |D| for a graph of the transient displacement $X_h(T)$. The result is shown in Screen 3.

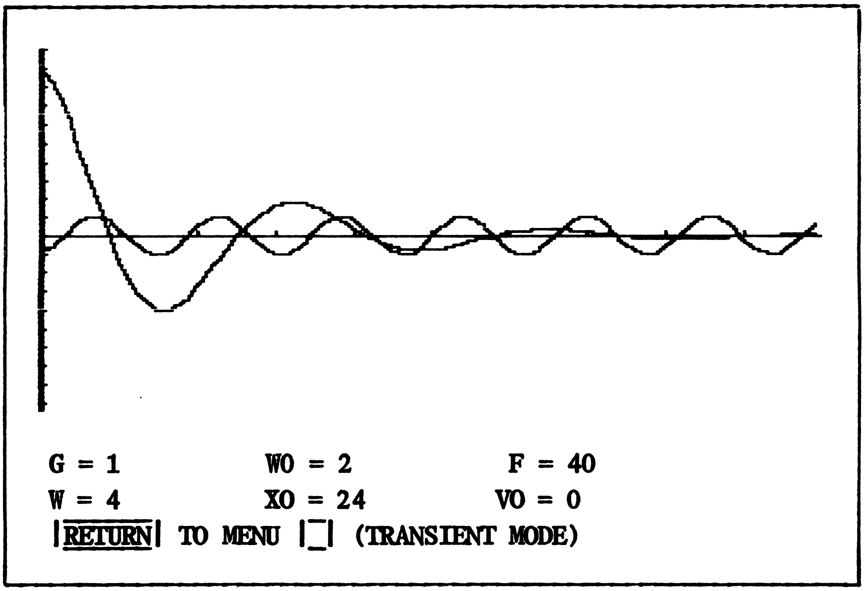
Next, press |N| |S| |D| to graph $X_p(T)$ (Screen 4).

To compare the graphs of $X_h(T)$ and $X_p(T)$ more closely we add the graph of $X_h(T)$ to Screen 4. Press |T| and then |D| without first clearing the screen. (See Screen 5.)

It is the sum $X(T) = X_h(T) + X(T)$ that is graphed in Screen 2. Note that neither $X_h(T)$ nor $X_p(T)$ alone satisfies the initial condition $X(0) = 24$ set forth for $X(T)$. The value of $X_h(0)$ is greater than 24, and $X_p(0)$ is negative. It is the sum of $X_h(0)$ and $X_p(0)$ that equals 24. When you are ready, add the graph of $X(T)$ to the display in Screen 5 by pressing |B| |D|.



Screen 4. The graph of $X_p(T)$, the particular or steady state part of $X(T)$.

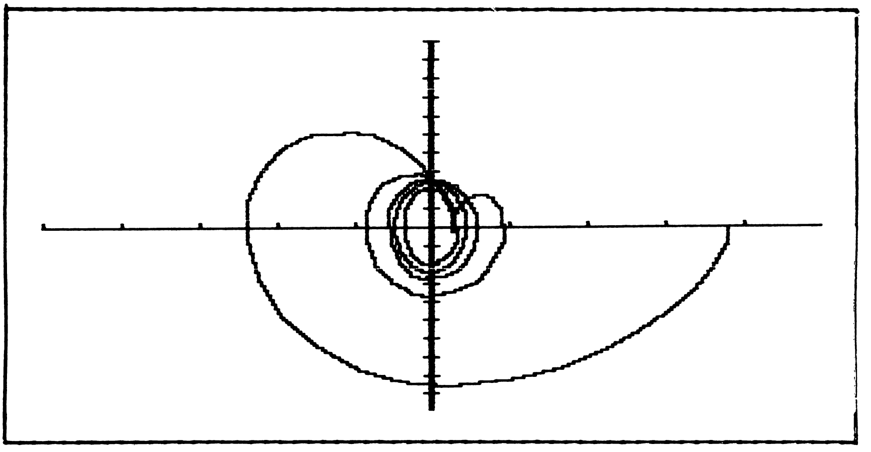


Screen 5. $X_h(T)$ and $X_p(T)$ together for $0 \leq T \leq 10$.

To see the phase plane, press |N| for a new screen. Then press |R| to rescale and enter VMAX = 50 and XMAX = 18, leaving the other settings as they are. Press |P|. The resulting graph of V(T) vs. X(T) is shown in Screen 6. If you wish, you may superimpose the graphs of X(T) and V(T) to correlate the information.

The phase plane plot is also available in the transient and steady state modes. For example, to see the graph of $\dot{X}_h(T)$ vs. $X_h(T)$, press |N| |T| |P|. This graph can also be correlated with the individual graphs of $X_h(T)$ and $\dot{X}_h(T)$.

To conclude the example, plot the energy E(T) and the velocity V(T) in a common display and note the relationship between the two. Pay particular attention to the energy curve when the graph of V(T) is approximately horizontal, and vice versa.



Screen 6. V(T) vs. X(T) for the driven oscillator, $0 \le T \le 10$. The path starts on the far right with X = 24, V = 0, swings wide, backs up momentarily, and then continues its clockwise motion to spiral in toward the limiting cycle that represents the steady state solution.

PROBLEMS

1. Rerun the text example with F = −40.
2. Rerun the text example (a) with G = .5, and (b) with G = .5 and F = −40.
3. Rerun the text example
   a) with G = .25, XMAX = 25, TMAX = 20
   b) with G = .25, XMAX = 25, TMAX = 20, and F = −40.
4. Set the program's main menu with
   G = 1, W0 = 2, F = 40, W = 4, X0 = 24, V0 = 0,
   as in the text example, and scale the display axes with XMAX = 30 and TMAX = 10. Then graph
   a) X(T), X_h(T), and X_p(T) together.
   b) V(T), V_h(T), and V_p(T) together.
   c) E(T), E_h(T), and E_p(T) together.

Problems 5–14 are about the following oscillator: A mass of 3.0 kg is suspended from a spring. The spring constant is 1200 N/m. Whenever the mass moves, it encounters a resistive force given by F = −bv, where b = 10 N sec/m and v is the velocity of the mass in m/sec. The mass is driven by sinusoidal force with W = 30/sec and an amplitude of 30 N.

5. Find the resonant frequency of the system.
6. Approximately how long does it take for the transient behavior of the system to disappear?
7. Find the amplitude of the forced vibration in the steady state.
8. Find out what happens to the amplitude of the forced vibration in the steady state for a succession of cases in which the driving frequency W lies (a) far below, (b) just below, (c) at, (d) just above, and (e) far above the resonant frequency WO.
9. In each part of Problem 4, examine the phase difference between the driving force and the amplitude of the forced vibration in the steady state.
10. Plot the amplitude of the forced vibration as a function of the frequency of the driving force divided by the resonant frequency, i.e., W/WO.
11. Plot the phase difference between the forced vibration and the driving force as a function of the frequency of the driving force divided by the resonant frequency, i.e., W/WO.
12. Examine the energy in the system as a function of time in (a) Problem 10 and (b) Problem 11.
13. Examine the phase plane diagram in (a) Problem 10 and (b) Problem 11.
14. Do the representations in Problems 10 and 11 contain the same information? If not, what are the differences?

Appendix A. Algebraic Notation

1. PURPOSE

This appendix describes some of the computer notation for algebraic operations.

2. COMPUTER FORMAT

Operation	Computer Expression	Textbook Notation
add	X + Y	$x + y$
subtract	X - Y	$x - y$
multiply	X * Y	xy, $x \cdot y$, or $x \times y$
divide	X / Y	x/y or $x \div y$
raise to power	X ^ Y	x^y

For example, press |5| |*| |X| |RETURN| to enter the product 5x and press |5| |*| |X| |+| |Y| |RETURN| to enter the expression 5x + y.

Always use an asterisk (*) for multiplication. The computer interprets XY (without the asterisk) as the name of a single variable and will report a syntax error if you attempt to enter 5Y.

Operations are carried out from left to right according to three "levels" of precedence:

Level 1. ^ (first precedence, executed first)
Level 2. *, / (executed next)
Level 3. +, - (executed last).

Examples

5 * 3 + 2 = 15 + 2 = 17
5 + 3 * 2 = 5 + 6 = 11
5 + 2 ^ 3 = 5 + 8 = 13
2 - 5 * 2 ^ 3 = 2 - 5 * 8 = 2 - 40 = -38.

Parentheses take precedence over all binary operations and may therefore be used to control the order of execution.

Examples

(2 + 5) * 2 ∧ 3 = 7 * 8 = 56
((2 + 5) * 2) ∧ 3 = (7 * 2) ∧ 3 = 14 ∧ 3 = 2,744
3 / (2 − 4) = 3 / (−2) = −1.5

Within any level of precedence, evaluation proceeds from left to right.

Examples

3 + 5 − 4 = 8 − 4 = 4
5 − 4 + 3 = 1 + 3 = 4
3 * 8 / 4 = 24 / 4 = 6
8 / 4 * 3 = 2 * 3 = 6

The last of these examples indicates a usage somewhat different from that on the printed page. In evaluating the textbook expression x/yz, you first multiply y by z, then divide. Resist the temptation to use X / Y * Z as the corresponding computer expression. The computer executes the operations in this expression from left to right because / and * share a precedence level. It therefore evaluates X / Y * Z as X / Y * Z = (X / Y) * Z.

To evaluate	Use
$\frac{x}{yz}$	X / Y / Z or X / (Y * Z)
$\frac{xy}{z}$	X * Y / Z, X / Z * Y, or (X * Y) / Z.

Division by zero is never possible because no expression of the form a/0 satisfies the definition of division. This point is especially important in computer operation, since the problem may be hidden in some ways. For example, a

particular application may involve evaluating the function

$$f(x) = (x^3 - 1)/(x^2 + 2x + 1)$$

at points equally spaced across the interval from −2 to 2. If care is not taken, the corresponding computer variable X may be assigned the value −1, and a "division by zero" error would result.

There are exceptions to the guidance on division by zero, however. Several Toolkit programs contain a subroutine that checks for an impending instruction to divide by zero, cancels any such instruction when it finds one, and takes an appropriate course of action in executing the graphics routines. Thus, for example, in using SUPER*GRAPHER, you may ask for a graph of f(x) = 1/x from −2 to 2, specify a number of plotting points that results in the assignment of 0 to the computer variable X, and still obtain a good display.

3. SPACES

You may include spaces in algebraic expressions (X + Y) or omit them (X+Y), as you please. The main reason for including spaces is legibility. However, spaces contribute to a formula's character count, and an expression that exceeds the number of characters allowed for entry into a Toolkit program menu may be sometimes shortened to an acceptable size by reducing the number of spaces.

Appendix B. Function Notation

1. PURPOSE

This appendix describes the format for entering formulas for commonly used mathematical functions into the computer.

2. FORMAT

In using Toolkit programs, you often need to enter formulas for mathematical functions. The following table lists expressions in BASIC for the functions that are evaluated directly by the computer's subroutines.

Mathematical Function	Computer Expression
$\sin x$	SIN(X)
$\cos x$	COS(X)
$\tan x$	TAN(X)
$\arctan x$	ATN(X)
$\sqrt{x}$	SQR(X)
$\lvert x \rvert$	ABS(X)
e^x, or $\exp x$	EXP(X)
$\ln x$	LOG(X)
$[x]$	INT(X)
$\operatorname{sgn} x$	SGN(X)

The greatest integer function $[x]$ is defined for all real numbers to be the greatest integer that is less than or equal to x. Thus, $[4.5] = 4$, $[4] = 4$, and $[-1.3] = -2$.

The signum function $\operatorname{sgn} x$ is defined by the rule

$$\operatorname{sgn} x = \begin{cases} -1 & \text{if } x < 0 \\ 0 & \text{if } x = 0. \\ 1 & \text{if } x > 0 \end{cases}$$

Thus, $\operatorname{sgn}(-3.5) = -1$, $\operatorname{sgn} 0 = 0$, and $\operatorname{sgn} 5 = 1$.

Examples

Mathematical Function	Computer Expression
sin 2x	SIN(2*X)
ln (x + y)	LOG(X + Y)
cos πx	COS(PI*X)
arctan(y/x) or $\tan^{-1}(y/x)$	ATN(Y/X)

Many other functions can be expressed in terms of the computer's BASIC functions:

Mathematical Function	Computer Expression
cot x	1/TAN(X)
sec x	1/COS(X)
csc x	1/SIN(X)
arc sin x	ATN(X)/SQR(1 - X*X)
arc cos x	PI/2 - ATN(X/SQR(1 - X*X))
arc cot x	PI/2 - ATN(X)
arc sec x	ATN(SQR(X*X - 1)) + (SGN(X) - 1) * PI/2
sinh x	(EXP(X) - EXP(-X))/2
cosh x	(EXP(X) + EXP(-X))/2
tanh x	1 - 2*EXP(-X)/(EXP(X) + EXP(-X))
coth x	1 + 2*EXP(-X)/(EXP(X) - EXP(-X))
sech x	2/(EXP(X) + EXP(-X))
arc sinh x	LOG(X + SQR(1 + X*X))
arc cosh x	LOG(X + SQR(X*X - 1))
arc tanh x	LOG((1 + X)/(1 - X))/2
arc coth x	LOG((X + 1)/(X - 1))/2
arc sech x	LOG(SQR(1 - X*X) + 1)/X
arc csch x	LOG(SGN(X)*SQR(1 + X*X) + 1)/X

The computer is always in radian mode: SIN(1.5) is evaluated as the sine of 1.5 radians and ATN(1) returns a decomal approximation of π/4.

Most Toolkit programs accept PI for π (press |P| |I|). It is usually not necessary to enter a decimal approximation. Some programs also accept E for e.

Answers to Selected Problems

A. SUPER*GRAPHER

Section 5: 1) 1 2) 2/3 3) 1/2 4) 0 5) 2/5
6) e = 2.71828... 7) 0 8) $\pi/2$

Section 6:
1. V takes on a maximum value of $C1C2^2/4$ at X = C2/2, a point whose location depends on C2 alone.
2. Maximum at X = C1/6
4. Maximum at X = 2C1/3
5. a and b: $-\sqrt{2} \leq F \leq \sqrt{2}$
   c) $-1/2 \leq F \leq 1/2$

J. ROOT FINDER

13. With X0 = 2, X1 = 4: K = 1 for B, K = 4 for S, K = 5 for R, K = 3 for N
14. $X^* = 2, 4, -.766664696$
15. At K = 11 the computer finds $X^* = 1.00024414$. As far as the computer is concerned, $(.00024414)^{11} = 0$.
16. The computation stops at K = 6 with F(1.0078152) = 0. As far as the computer is concerned, the root has been found. With TOL = 1E-100 the computation will still stop at K = 6.
17. $X^* = -1.53208889, -.347296335, 1.87938524$
18. $X^* = .630115396, 2.57327196$
19. $X^* = 0, .868876851, 1$
20. $X^* = \pm 1.306562965, \pm .5411961001$

M. INTEGRAL EVALUATOR

1) −4 2) 2 3) 8.4 4) 1.14779358 5) 1.57074898
6) 3.24130926 7) 0 (Computed values are around 10^{-8}) 8) 1
9) .970753907 10) 2.54517744 11) 1.71828183
12) 1.22619117 13) 26.9738382 14) 1.13197175
15) 1.17520119 16) .077520710 17) −.847382017
18) 1.14915123 19) −.25 20) 656.528364 21) .915965595
22) 2.9253 (to four places) 23) .1963495408 24) 1.3114425

P. CONIC SECTIONS

The answers to Problems 6 – 10 will vary.

6. Among the possibilities are: a) Up 1 b) Up 1, rotate $+45^o$ c) Up 1, rotate $+90^o$ d) Down 1, rotate -45^o
7. Among the possibilities are: a) Left 2 b) Left 2, down 2 c) Left 2, down 2, rotate $+45^o$ d) Left 2, down 2, rotate $+90^o$. Try left 2, up 2 as well.
8. Among the possibilities are: a) Up 3 b) Up 3, rotate ± 180 c) Down 3 d) Rotate $+45^o$
9. Among the possibilities are: a) Rotate $+90^o$ b) Rotate $+15^o$ c) Right 1, up 1 d) Right 2, down 3
10 Among the possibilities are: a) Rotate $+90^o$ b) Rotate $+90^o$, right 1 c) Rotate $+90^o$, left 10, up 1 d) Rotate 180^o, right 1
11. a,b,d, and e: no change
    c) Changes the sign of one side of the equation.
12. All: Interchanges x and y. In (b), this means no change in the equation.

The motions in Answers 13 – 17 are not the only ones to work.

13. Hyperbola, left 7
14. Line, down 5, rotate $+135^o$
15. Ellipse, right 3
16. Parabola, left 11
17. Parabola, UNIT = .1, up 1

Q. SEQUENCES AND SERIES

16. a) $\pi/2$ b) $2\pi + \pi/2$ c) $\pi/2$
17. a) 2 b) 1.5 c) 4.99994291 d) .909090909

U. DOUBLE INTEGRAL EVALUATOR

3) 1 4) .5 5) 5/6 6) 16/3 7) 1.875 8) 1/6 9) .60347448 10) π 11) 9 12) .199788200 13) .287828705 14) 6.27657703 15) .233207814 16) .247619048

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